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A NONLINEAR THEORY FOR THE BENDING, BUCKLING, AND VIBRATIONS OF CONICAL SHELLS

By

Dror Bendavid and J. Mayers
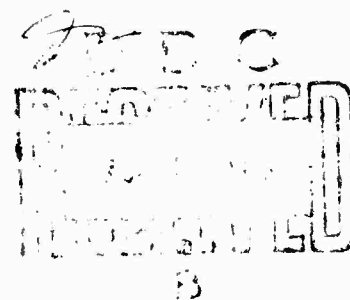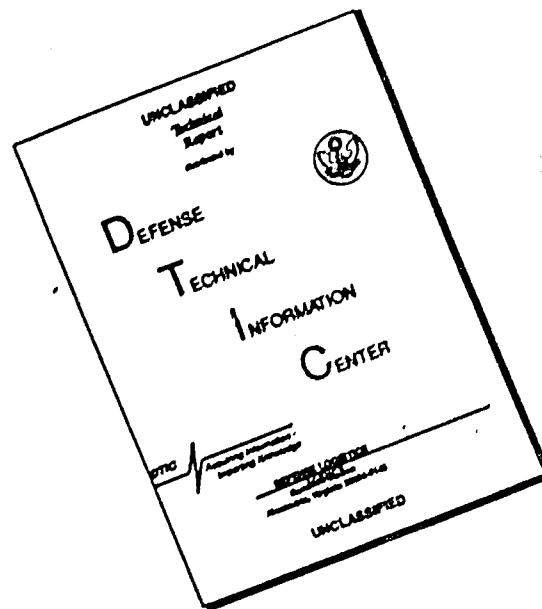
June 1970

U. S. ARMY AVIATION MATERIEL LABORATORIES
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This program was carried out under Contract DAAJ02-68-C-0035 with Stanford University.

The data contained in this report are the result of research conducted to develop a nonlinear theory for the bending, buckling, and vibration of conical shells. The equations of motion and associated boundary conditions are presented.

This report has been reviewed by the U.S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

A NONLINEAR THEORY FOR THE BENDING, BUCKLING, AND
VIBRATIONS OF CONICAL SHELLS

By

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FORT EUSTIS, VIRGINIA

SUMMARY

Equations of motion and associated boundary conditions are developed for the general nonlinear vibrational behavior of thin conical shells. The theory is based upon nonlinear strain-displacement relations deduced for a conical shell from those derived by Sanders for thin shells of compound curvature. Equations for the bending, buckling, and postbuckling of conical shells under arbitrary loads are developed also and are shown to reduce to equations based on more simplified theories for both conical and circular cylindrical shells and circular flat plates. Various solution approaches to the nonlinear conical shell vibration problem are examined, and a new numerical method of solution is proposed and discussed.

FOREWORD

The work reported herein constitutes a portion of a continuing effort being undertaken at Stanford University for the U. S. Army Aviation Materiel Laboratories under Contract DAAJ02-68-C-0035 (Task 1F162204A17002) to establish accurate theoretical prediction capability for the static and dynamic behavior of aircraft structural components utilizing both conventional and unconventional materials. Predecessor contracts supported investigations which led, in part, to the results presented in References 18, 28, 29, 32, and 33.

LIST OF SYMBOLS

$A_i(t)$ Time-dependent radial displacement coefficients, in.

D Bending rigidity $= \dfrac{Eh^3}{12(1-\nu^2)}$, lb-in.

E Young's modulus of elasticity, lb/in.^{2}

F Stress function, lb

h Shell thickness, in.

i Integer

L Length of circular cylindrical shell, in.

ℓ,m,n,ℓ',m',n' Integers

$M_x, M_\varphi, M_{x\varphi}$ Bending and twisting moments, lb-in./in.

$\overline{M}_x, \overline{M}_\varphi, \overline{M}_{x\varphi}$ Applied edge moments, lb-in./in.

$N_x, N_\varphi, N_{x\varphi}$ Forces in shell middle surface, lb/in.

$\overline{N}_x, \overline{N}_\varphi, \overline{N}_{x\varphi}$ Applied edge forces, lb/in.

q_x, q_φ, q_z Applied surface loads, lb/in.^{2}

Q_x, Q_φ Shearing forces, lb/in.

$\overline{Q}_x, \overline{Q}_\varphi$ Applied edge shearing forces, lb/in.

R Radius of circular cylindrical shell, in.

R_1, R_2 Principal radii of curvature of thin shell of arbitrary shape, in.

t Time, sec

T Kinetic energy, lb-in.

u Component of the midsurface displacement in the axial direction, in.

U Total strain energy, lb-in.

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U_1, U_2	Displacements tangential to middle surface of thin shell of arbitrary shape, in.
U_b	Strain energy due to bending, lb-in.
U_m	Strain energy due to middle-surface stretching, lb-in.
U_T	Total potential energy in the presence of applied loads, lb-in.
v	Component of the midsurface displacement in the tangential or circumferential direction, in.
V_b	Potential of applied edge moments and shears, lb-in.
V_L	Potential of applied surface loads, lb-in.
V_m	Potential of applied edge loads, lb-in.
w	Component of the midsurface displacement in the radial direction, in.
W	Displacement normal to midsurface of a thin shell of arbitrary shape, in.
x	Axial coordinate on middle surface, in.
x_1	Distance of the topmost cross section of a truncated cone from the vertex, measured along a generator, in.
x_2	Distance of the base of a truncated cone from the vertex, measured along a generator, in.
z	Radial coordinate, normal to middle surface, in.
α	Cone semi-vertex angle, rad
α_1, α_2	Coefficients in metric form of middle surface of thin shell of arbitrary shape
$\epsilon_{11}, \epsilon_{22}, \epsilon_{12}$	Middle-surface strains for thin shell of arbitrary shape, in./in.
$\epsilon_x, \epsilon_\varphi, \gamma_{x\varphi}$	Middle-surface strains, in./in.
$\varkappa_{11}, \varkappa_{22}, \varkappa_{12}$	Middle-surface curvatures for thin shell of arbitrary shape, 1/in.
$\varkappa_x, \varkappa_\varphi, \varkappa_{x\varphi}$	Middle-surface curvatures, 1/in.
ν	Poisson's ratio

ξ_1, ξ_2	Coordinates on middle surface of thin shell of arbitrary shape
$\vdash$	Mass density of shell, $lb\text{-}sec^2/ft^4$
φ	Circumferential coordinate on middle surface, rad
φ_1, φ_2	Specific generators located at $\varphi = \varphi_1$ and $\varphi = \varphi_2$, rad
Φ_1, Φ_2	Rotations, rad
Φ_n	Rotation about the normal to shell middle surface, rad

Operator Symbols

$$\nabla_c^2(\ ) = (\ )_{,xx} + (\ )_{,x}/x + (\ )_{,\varphi\varphi}/x^2\sin^2\alpha$$

$$\nabla_c^4(\ ) = \nabla_c^2[\nabla_c^2(\ )]$$

$$\nabla^4(\ ) = \nabla^2[\nabla^2(\ )] = \nabla^2[(\ )_{,xx} + (\ )_{,yy}]$$

Subscript

c	Denotes conical shell

INTRODUCTION

All investigations of the free vibrations of conical shells to date have used linear theory, mostly based upon Love's first approximation theory for thin shells[1]. Some of the notable earlier studies are those by Strutt[2], Federhofer[3], Goldberg[4], Grigolyuk[5], Shuman[6], Saunders, Wisniewski, and Paslay[7], and Garnet[8]. The last-named author modified the theory to take into account transverse shear and rotatory inertia effects.

With the development in the late 1950s of the more sophisticated theory for conical shells (developed specifically for buckling problems) by Seide[9] and Singer[10], free vibration studies employing this theory were later carried out by Seide[11] and Weingarten[12]. These studies were based on the linear strain-displacement relations given by Love's first approximation with circumferential displacements in the curvature terms neglected and with midsurface inertia terms omitted. Results of these studies correlated well with experimental data for vibrating conical shells exhibiting a large number of circumferential waves. This was expected, since neglect of the midsurface inertias was shown by Reissner[13] to be valid only when a large number of circumferential waves is present.

In two other studies, notably those by Chen[14] and Weingarten[15], the authors retained the circumferential terms in the curvature expressions and also included the previously neglected midsurface inertia terms. Chen[14] used a Lagrange-equation approach in conjunction with the Rayleigh-Ritz method for a conical shell with classical simple supports. Again, correlation with experimental data was good for higher mode numbers, whereas results for a small number of waves correlated poorly. Weingarten[15] used Saunders'[16] linear shell equations in conjunction with a finite-difference scheme introduced by Budiansky and Radkowski[17] for static analysis of shells. The effects of different boundary conditions upon the vibrations were assessed, and it was shown that at low wave numbers the boundary conditions have a marked effect upon the frequencies.

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This was also shown in the earlier work of Seide[11] for two different
types of circumferential boundary restraints of simply supported conical
shells.

To the best knowledge of the authors, no investigation of the vibrations
of thin conical shells using nonlinear theory has been carried out to
date. It is therefore the purpose of the present work to develop the
governing equations and the associated boundary conditions for the non-
linear vibrational behavior of a thin truncated conical shell and to
examine and discuss various possible approaches for solving these more
accurate but complex equations.

Recent investigations of the related problem of nonlinear vibrations of a
circular cylindrical shell have produced some interesting results. In a
major work on the subject, in which midsurface inertia terms are neglected,
Mayers and Wrenn[18] criticize previous solutions[19,20,21,22] based upon the
well-known von Kármán-Donnell strain-displacement relations. The authors
carry out a new solution for the classical equations which removes the
basis for criticism and discloses the existence of a nonperiodic vibration
behavior, a phenomenon not considered previously. Solutions for a small
number of circumferential waves are obtained also using the more accurate
strain-displacement relations derived by Sanders[23].

In the present investigation, the governing equations for the nonlinear
vibrational behavior of a conical shell are developed using nonlinear
strain-displacement relations for a conical shell deduced from those
derived by Sanders[23] for a shell of arbitrary shape. The total strain
and kinetic energies of the shell are developed for application in
Hamilton's variational principle; the governing differential equations of
motion and associated boundary conditions are obtained therefrom. As a
special case, these equations are shown to reduce to the so-called "clas-
sical" equations of conical shells; namely, those for which the circum-
ferential terms in the curvature and nonlinear midsurface strain expres-
sions are omitted and the midsurface inertia terms are neglected.

Although the present work deals with the nonlinear vibrations behavior

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of a conical shell, with little added effort the governing equations
and associated boundary conditions for the nonlinear behavior of a
conical shell under arbitrary static and dynamic loads are obtained also.
These equations are valid for either forced nonlinear vibration problems,
nonlinear vibration of a prestressed shell, or prebuckling and postbuckling
nonlinear behavior of an arbitrarily loaded shell. These extended
equations are presented in Appendix I. Linearized stability equations
and boundary conditions using the nonlinear terms developed in this
study are also presented in Appendix II.

The equations derived throughout this investigation are quite general in
that they reduce (by a set of simple substitutions) to corresponding
equations for circular cylindrical shells and circular flat plates. These
equations are presented in Appendixes III and IV, respectively, and they are
compared with equations derived in previous studies.

Lastly, in attempting to solve the nonlinear vibration problem for a
conical shell, related work done for the limiting case of the circular
cylindrical shell is examined.

The first known investigation of the nonlinear vibrations of a circular
cylindrical shell was carried out by Reissner[24] using the classical
equations (i.e., with midsurface inertia neglected). For linear
vibrations, Reissner assumed that the radial displacement function may be
represented as a chessboard pattern. This assumption was validated by
previous and recent experimental work as reviewed in Reference 18. For
nonlinear vibrations, Reissner assumed that the nonlinearity has a more
pronounced effect on the arbitrary time function, which modifies the
choice of deflected shape, than on the deflected shape itself; conse-
quently, Reissner also used the chessboard pattern for the nonlinear
vibration solution. However, as pointed out by Evensen[21], the use of the
chessboard pattern leads to a circumferential displacement that is not a
periodic function of the circumferential coordinate.

Another solution, which like Reissner's did not satisfy the periodicity
condition, was presented by Chu[19]. In a later study by Nowinski[20], the
periodicity was taken into account; but as a consequence (as noted by

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Evensen[21]), a nonzero radial displacement occurred at the end of the shell, thus violating the assumed geometric boundary conditions of free support. All results obtained in the foregoing references were based upon periodic motion, except those of Mayers and Wrenn[18].

The nonlinear behavior experienced by a circular cylindrical shell should be experienced as well (perhaps in a slightly modified form) by a conical shell. Consequently, in SUGGESTED METHODS OF SOLUTION, extension to a conical shell of the methods of solution used by the cylindrical shell investigators is discussed and an alternative solution based upon a modified-Reissner variational principle approach (as introduced by Mayers et al.[28,29]) is also suggested. Finally, a new direct numerical scheme for solution is proposed and demonstrated.

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STRAIN-DISPLACEMENT RELATIONS

The nonlinear strain-displacement relations employed in this study are deduced from those derived by Sanders[23] for thin shells of arbitrary shape. In the middle surface, the strains are given by

$$\epsilon_{11} = \frac{1}{\alpha_1 \alpha_2}[\alpha_2 U_{1,\xi_1} + \alpha_{1,\xi_2} U_2] - \frac{W}{R_1} + \frac{1}{2}(\Phi_1^2 + \Phi_n^2)$$

$$\epsilon_{22} = \frac{1}{\alpha_1 \alpha_2}[\alpha_1 U_{2,\xi_2} + \alpha_{2,\xi_1} U_1] - \frac{W}{R_2} + \frac{1}{2}(\Phi_2^2 + \Phi_n^2) \tag{1}$$

$$\epsilon_{12} = \frac{1}{2\alpha_1 \alpha_2}[\alpha_2 U_{2,\xi_1} + \alpha_1 U_{1,\xi_2} - U_1 \alpha_{1,\xi_2} - U_2 \alpha_{2,\xi_1}] + \frac{1}{2}\Phi_1 \Phi_2$$

where the rotations Φ_1 and Φ_2 are

$$\Phi_1 = \frac{U_1}{R_1} + \frac{1}{\alpha_1} W_{,\xi_1}$$

$$\Phi_2 = \frac{U_2}{R_2} + \frac{1}{\alpha_2} W_{,\xi_2} \tag{2}$$

and the rotation about the normal to the middle surface is

$$\Phi_n = \frac{1}{2\alpha_1 \alpha_2}[(\alpha_2 U_2)_{,\xi_1} - (\alpha_1 U_1)_{,\xi_2}] \tag{3}$$

The general curvature-displacement relations used by Sanders[23] are given by

$$\kappa_{11} = \frac{1}{\alpha_1 \alpha_2}[\Phi_{1,\xi_1}\alpha_2 + \Phi_2 \alpha_{1,\xi_2}]$$

$$\kappa_{22} = \frac{1}{\alpha_1 \alpha_2}[\Phi_{2,\xi_2}\alpha_1 + \Phi_1 \alpha_{2,\xi_1}] \tag{4}$$

$$\kappa_{12} = \frac{1}{2\alpha_1 \alpha_2}[\Phi_{2,\xi_1}\alpha_2 + \Phi_{1,\xi_2}\alpha_1 - \Phi_1 \alpha_{1,\xi_2} - \Phi_2 \alpha_{2,\xi_1}] + \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\Phi_n$$

Equations (1-4) are valid under the following assumptions:

1. The Kirchoff-Love hypothesis holds; that is, a straight line segment that is perpendicular to the middle surface of the

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deformed shell remains perpendicular to the deformed middle surface while undergoing negligible strain relative to its original length

2. The strains and rotations of line elements on the shell mid-surface induced by the deformation remain small in comparison to unity, although the components of the displacement are not necessarily "small".

Thus, the total strains in terms of displacements are

$$\epsilon_{11}^* = \epsilon_{11} + z\,\varkappa_{11}$$

$$\epsilon_{22}^* = \epsilon_{22} + z\,\varkappa_{22}$$

$$\epsilon_{12}^* = \epsilon_{12} + 2z\,\varkappa_{12}$$

By choosing the coordinates for a truncated conical shell as shown in Figure 1, the general parameters appearing in equations (1)-(4) may be written as

$$\xi_1 = x \qquad \text{axial coordinate along a generator}$$

$$\xi_2 = \varphi \qquad \text{circumferential coordinate}$$

$$U_1 = u \qquad \text{displacement component in x direction}$$

$$U_2 = v \qquad \text{displacement component in } \varphi \text{ direction}$$

$$W = w \qquad \text{displacement component in radial direction, positive inward}$$

Also,

$$\alpha_1 = 1$$

$$\alpha_2 = x\sin\alpha$$

$$\frac{1}{R_1} = 0 \tag{5}$$

$$\frac{1}{R_2} = \frac{\cos\alpha}{x\sin\alpha}$$

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and

$$\epsilon_{11} = \epsilon_x$$

$$\epsilon_{22} = \epsilon_\varphi$$

$$\epsilon_{12} = \frac{1}{2}\gamma_{x\varphi}$$

$$\varkappa_{11} = \varkappa_x$$

$$\varkappa_{22} = \varkappa_\varphi$$

$$\varkappa_{12} = \varkappa_{x\varphi}$$

(6)

The rotations given by equations (2) and (3) consequently reduce to

$$\Phi_{1_c} = w,_x$$

$$\Phi_{2_c} = \frac{v\cos\alpha + w,_\varphi}{x\sin\alpha}$$

(7)

$$\Phi_{n_c} = \frac{1}{2}\left(\frac{v}{x} + v,_x - \frac{u,_\varphi}{x\sin\alpha}\right)$$

and the nonlinear strain and curvature relations for a conical shell become

$$\epsilon_x = u,_x + \frac{1}{2}w,_x^2 + \frac{1}{2}\Phi_{n_c}^2$$

$$\epsilon_\varphi = \frac{u}{x} + \frac{v,_\varphi - w\cos\alpha}{x\sin\alpha} + \frac{1}{2x^2\sin^2\alpha}(w,_\varphi^2 + v^2\ \widetilde{\cos}^2\ \alpha + 2vw,_\varphi\cos\alpha)$$

$$+ \frac{1}{2}\Phi_{n_c}^2$$

(8)

$$\gamma_{x\varphi} = v,_x + \frac{u,_\varphi}{x\sin\alpha} - \frac{v}{x} + \frac{w,_x}{x\sin\alpha}(w,_\varphi + v\cos\alpha)$$

$$\varkappa_x = w,_{xx}$$

$$\varkappa_\varphi = \frac{w,_{\varphi\varphi} + v,_\varphi \cos \alpha}{x^2 \sin^2 \alpha} + \frac{w,_x}{x} \tag{9}$$

$$\varkappa_{x\varphi} = \frac{w,_{x\varphi} + \frac{3}{4} v,_x \cos \alpha}{x \sin \alpha} - \frac{w,_\varphi + \frac{3}{4} v \cos \alpha}{x^2 \sin \alpha} - \frac{u,_\varphi \cos \alpha}{4x^2 \sin^2 \alpha}$$

Equations (8) and (9) appear as a displacement formulation in terms of a tangential v-displacement notation. However, they may also be written in terms of a circumferential v-displacement notation to effect a convenient simplification as shown for a circular cylindrical shell by Mayers et al.[18] Throughout this report, the terms which vanish when circumferential v-displacements are employed ɛ traced by a ~ superscript, as in the second of equations (8).

At this point another assumption is made; that is, the angles of rotation Φ_{n_c} about the normals to the shell midsurface are assumed to remain negligibly smaller than the rotations (Φ_{1_c} and Φ_{2_c}) out of the midsurface throughout the deformation. Therefore,

$$\Phi_{n_c} = 0 \tag{10}$$

and equations (8) and (9) reduce further to

$$\epsilon_x = u,_x + \frac{1}{2} w,_x^2$$

$$\epsilon_\varphi = \frac{u}{x} + \frac{v,_\varphi - w \cos \alpha}{x \sin \alpha} + \frac{1}{2x^2 \sin^2 \alpha}(w,_\varphi^2 + v^2 \overset{*}{\underset{\sim}{\cos}}^2 \alpha + 2vw,_\varphi \overset{*}{\cos} \alpha)$$

$$\gamma_{x\varphi} = v,_x + \frac{u,_\varphi}{x \sin \alpha} - \frac{v}{x} + \frac{w,_x}{x \sin \alpha}(w,_\varphi + v \overset{*}{\cos} \alpha) \tag{11}$$

$$\varkappa_x = w,_{xx}$$

$$\varkappa_\varphi = \frac{w,_{\varphi\varphi} + v,_\varphi \overset{*}{\cos} \alpha}{x^2 \sin^2 \alpha} + \frac{w,_x}{x} \qquad \text{(continued)}$$

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$$\kappa_{x\varphi} = \frac{w,_{x\varphi} + \frac{1}{2} v,_{x}^{*} \cos \alpha}{x \sin \alpha} - \frac{w,_{\varphi} + v^{*} \cos \alpha}{x^2 \sin \alpha} \tag{12}$$

As noted in the introduction, previous authors studying conical shell behavior, notably Seide[9,11] and Singer[10], have employed simplified formulations of equations (11) and (12) by neglecting the circumferential displacement component and its derivatives in the curvature-displacement relations. These terms were also neglected in the nonlinear strain-displacement relations.*

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*Throughout this report, the Seide formulation[9] will be referred to as "classical" conical shell theory, and the terms which are neglected in said formulation will be traced by a (*) superscript, as in equations (11) and (12).

TOTAL POTENTIAL ENERGY

The total potential energy U_T of the shell in the presence of applied external loads may be expressed as

$$U_T = U_m + U_b + V_L + V_m + V_b \tag{13}$$

where U_m is the strain energy due to middle-surface stretching, U_b is the strain energy due to bending, V_L is the potential of the applied surface loads, V_m is the potential of the applied edge loads, and V_b is the potential of the applied edge moments and shears.

An analysis of the nonlinear behavior of a conical shell in the presence of external loads is presented in Appendix I; in Appendix II, the resulting general stability equations of a conical shell are discussed. In this section, however, external loads are not considered; instead, the equations of motion for the nonlinear free vibrations of a conical shell are developed.

In the absence of applied external loads, the total potential energy of the shell is given by the strain energy

$$U = U_m + U_b \tag{14}$$

The strain energy due to middle-surface stretching may be expressed as

$$U_m = \frac{1}{2} \iint_A (N_x \epsilon_x + N_\varphi \epsilon_\varphi + N_{x\varphi} \gamma_{x\varphi}) dA \tag{15}$$

where N_x, N_φ, and $N_{x\varphi}$ are the middle-surface forces shown in the figure on page 11; ϵ_x, ϵ_φ, and $\gamma_{x\varphi}$ are the middle-surface strains; and A is the surface area.

Likewise, the strain energy due to bending may be expressed as

$$U_b = -\frac{1}{2} \iint_A (M_x \kappa_x + M_\varphi \kappa_\varphi - 2M_{x\varphi} \kappa_{x\varphi}) dA \tag{16}$$

where M_x, M_φ, and $M_{x\varphi}$ are the bending and twisting moments shown in the figure on page 11, and κ_x, κ_φ, and $\kappa_{x\varphi}$ are the middle-surface curvatures.

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Notation

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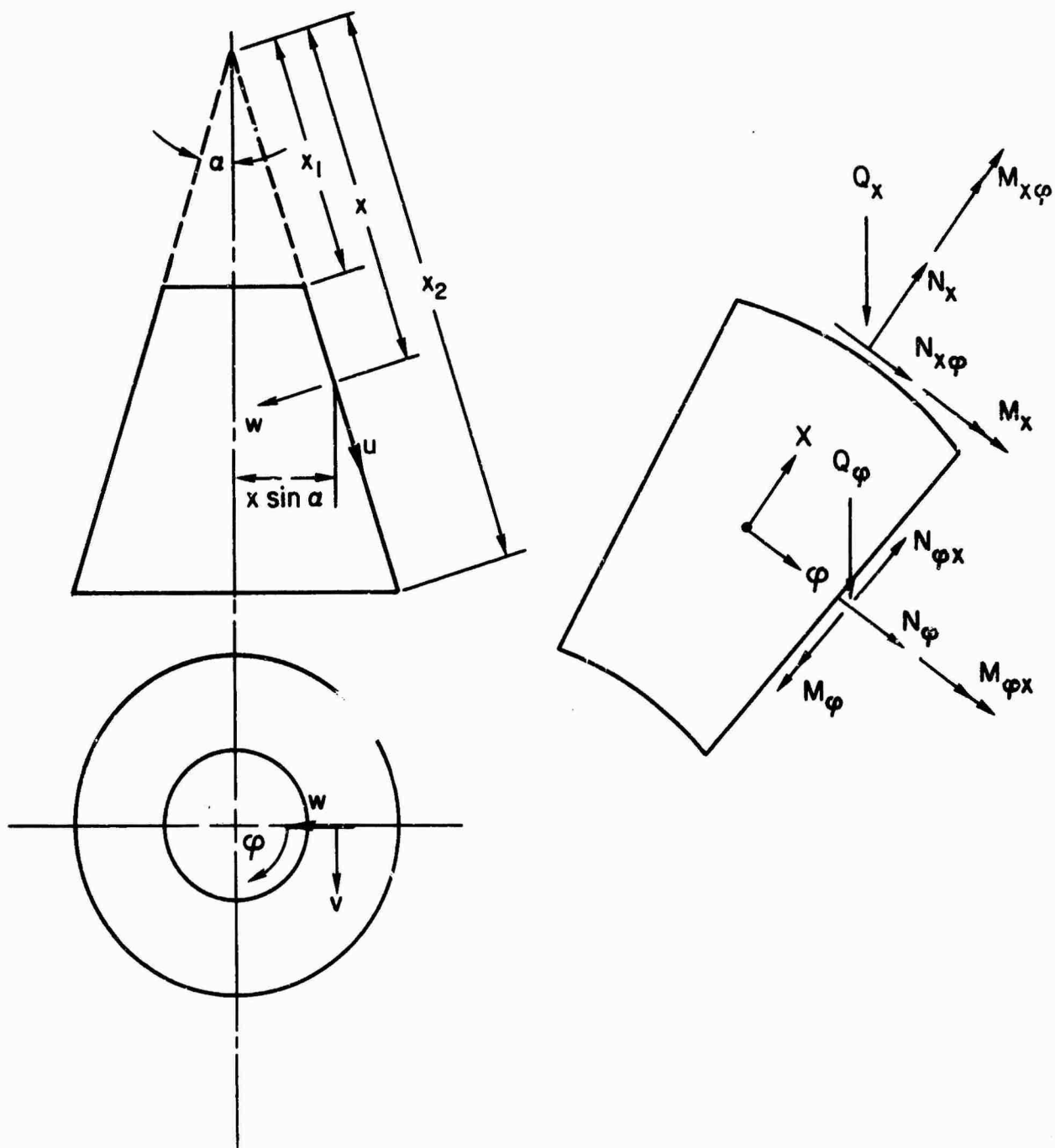
The forces and moments may be expressed in terms of the strains and curvatures, respectively, through the linear constitutive relations given by

$$N_x = \frac{Eh}{1 - \nu^2}(\epsilon_x + \nu\epsilon_\varphi)$$

$$N_\varphi = \frac{Eh}{1 - \nu^2}(\epsilon_\varphi + \nu\epsilon_x)$$

$$N_{x\varphi} = \frac{Eh}{2(1 + \nu)} \gamma_{x\varphi}$$

$$\qquad (17)$$

$$M_x = -D(\varkappa_x + \nu\varkappa_\varphi)$$

$$M_\varphi = -D(\varkappa_\varphi + \nu\varkappa_x)$$

$$M_{x\varphi} = -M_{\varphi x} = D(1 - \nu)\varkappa_{x\varphi}$$

$$\qquad (18)$$

where

$$D = \frac{Eh^3}{12(1 - \nu^2)} \qquad (19)$$

Substitution of equations (17) and (18) into equations (15) and (16) yields

$$U_m = \frac{Eh}{2(1 - \nu^2)} \iint_A [(\epsilon_x + \epsilon_\varphi)^2 - 2(1 - \nu)(\epsilon_x\epsilon_\varphi - \frac{1}{4} \gamma_{x\varphi}^2)]dA \qquad (20)$$

$$U_b = \frac{D}{2} \iint_A [(\varkappa_x + \varkappa_\varphi)^2 - 2(1 - \nu)(\varkappa_x\varkappa_\varphi - \varkappa_{x\varphi}^2)]dA \qquad (21)$$

The use of the relationship

$$dA = x \sin \alpha \, dx \, d\varphi \qquad (22)$$

for a conical shell, in conjunction with equations (11), (12), (14), (20), and (21), yields the expression for the strain energy in terms of displacements.

$$U = \frac{Eh}{2(1-\nu^2)} \int_0^{2\pi} \int_{x_1}^{x_2} \left\{ (u,_x + \frac{1}{2} w,_x^2)^2 + \left[\frac{u}{x} + \frac{v,_\varphi - w \cos \alpha}{x \sin \alpha} - \right. \right.$$

(continued)

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$$+ \frac{1}{2x^2 \sin^2 \alpha}(w,_\varphi^2 + v^2 \overset{\ast}{\underset{\sim}{\cos}}{}^2 \alpha + 2vw,_\varphi \overset{\ast}{\cos} \alpha)\Bigg]^2$$

$$+ 2\nu(u,_x + \frac{1}{2} w,_x^2)\Bigg[\frac{u}{x} + \frac{v,_\varphi - w \cos \alpha}{x \sin \alpha} + \frac{1}{2x^2 \sin^2 \alpha}(w,_\varphi^2 + v^2 \overset{\ast}{\underset{\sim}{\cos}}{}^2 \varphi$$

$$+ 2vw,_\varphi^{\ast} \cos \alpha)\Bigg] + \frac{1-\nu}{2}\Bigg[v,_x + \frac{u,_\varphi}{x \sin \alpha} - \frac{v}{x} + \frac{w,_x}{x \sin \alpha}(w,_\varphi + v \overset{\ast}{\cos} \alpha)\Bigg]^2\Bigg\}\cdot$$

$$x \sin \alpha \, dx \, d\varphi + \frac{D}{2} \int_0^{2\pi} \int_{x_1}^{x_2} \Bigg\{w,_{xx}^2 + \left(\frac{w,_{\varphi\varphi} + v,_\varphi \overset{\ast}{\cos} \alpha}{x^2 \sin^2 \alpha} + \frac{w,_x}{x}\right)^2$$

$$+ 2\nu w,_{xx} \left(\frac{w,_{\varphi\varphi} + v,_{\varphi} \overset{\ast}{\cos} \alpha}{x^2 \sin^2 \alpha} + \frac{w,_x}{x}\right) + 2(1-\nu)\left(\frac{w,_{x\varphi} + \frac{1}{2} v,_x \overset{\ast}{\cos} \alpha}{x \sin \alpha} - \right.$$

$$\left. - \frac{w,_\varphi + v \overset{\ast}{\cos} \alpha}{x^2 \sin \alpha}\right)^2\Bigg\} x \sin \alpha \, dx \, d\varphi \qquad (23)$$

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KINETIC ENERGY

The kinetic energy of the shell is the sum of the kinetic energies
associated with the axial, tangential and radial velocities; that is,

$$T = \frac{1}{2} \rho h \int_{\varphi_1}^{\varphi_2} \int_{x_1}^{x_2} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) x \sin \alpha \, dx \, d\varphi \qquad (24)$$

where φ_1 and φ_2 are taken from $\varphi_1 = 0$ to $\varphi_2 = 2\pi$ for the entire
shell.

VARIATIONAL PRINCIPLE

The application of Hamilton's principle requires that the simultaneous first-order change in the Lagrangian $(\mathcal{L} = T - U)$, integrated over a specified time interval with respect to admissible variations in the degrees of freedom characterizing the state of strain (namely, u, v, and w), must vanish; therefore,

$$
\left.\begin{aligned}
\delta_u \int_{t_1}^{t_2} (T - U)dt &= 0 \\[2em]
\delta_v \int_{t_1}^{t_2} (T - U)dt &= 0 \\[2em]
\delta_w \int_{t_1}^{t_2} (T - U)dt &= 0
\end{aligned}\right\} \tag{25}
$$

or

$$
\left.\begin{aligned}
\int_t \delta_u T \, dt - \int_t \delta_u U \, dt &= 0 \\[2em]
\int_t \delta_v T \, dt - \int_t \delta_v U \, dt &= 0 \\[2em]
\int_t \delta_w T \, dt - \int_t \delta_w U \, dt &= 0
\end{aligned}\right\} \tag{26}
$$

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DIFFERENTIAL EQUATIONS OF MOTION

Appropriate variation of the kinetic energy [equation (24)] yields

$$\delta_u T = \rho h \iint\limits_{\varphi\ x} \dot{u}\delta\dot{u}\ x\ \sin\alpha\ dxd\varphi$$

$$\delta_v T = \rho h \iint\limits_{\varphi\ x} \dot{v}\delta\dot{v}\ x\ \sin\alpha\ dxd\varphi \tag{27}$$

$$\delta_w T = \rho h \iint\limits_{\varphi\ x} \dot{w}\delta\dot{w}\ x\ \sin\alpha\ dxd\varphi$$

whereas integration with respect to time gives

$$\int_t \delta_u T\ dt = \rho h \iint\limits_{\varphi\ x} [\dot{u}\delta u]_{t_1}^{t_2}\ x\ \sin\alpha\ dxd\varphi - \rho h \iiint\limits_{t\ \varphi\ x} x\ \sin\alpha\ \ddot{u}\delta u\ dxd\varphi dt$$

$$\int_t \delta_v T\ dt = \rho h \iint\limits_{\varphi\ x} [\dot{v}\delta v]_{t_1}^{t_2}\ x\ \sin\alpha\ dxd\varphi - \rho h \iiint\limits_{t\ \varphi\ x} x\ \sin\alpha\ \ddot{v}\delta v\ dxd\varphi dt \tag{28}$$

$$\int_t \delta_w T\ dt = \rho h \iint\limits_{\varphi\ x} [\dot{w}\delta w]_{t_1}^{t_2}\ x\ \sin\alpha\ dxd\varphi - \rho h \iiint\limits_{t\ \varphi\ x} x\ \sin\alpha\ \ddot{w}\delta w\ dxd\varphi dt$$

The variation of the total strain energy U is

$$\delta U = \delta U_m + \delta U_b \tag{29}$$

where $\delta(\ ) = \delta_{u,v,w}(\ )$. Substitution of equations (15) and (16) into equation (29) yields

$$\delta U = \frac{1}{2} \iint\limits_{A} [(N_x \delta\epsilon_x + \epsilon_x \delta N_x) + (N_\varphi \delta\epsilon_\varphi + \epsilon_\varphi \delta N_\varphi) + (N_{x\varphi} \delta\gamma_{x\varphi} + \gamma_{x\varphi} \delta N_{x\varphi})]dA$$

(continued)

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$$- \frac{1}{2} \iint\limits_{A} [(M_x \delta \varkappa_x + \varkappa_x \delta M_x) + (M_\varphi \delta \varkappa_\varphi + \varkappa_\varphi \delta M_\varphi) - 2(M_{x\varphi} \delta \varkappa_{x\varphi} + \varkappa_{x\varphi} \delta M_{x\varphi})] dA \tag{30}$$

Now, for a linearly elastic material the following relations hold for all force and moment components.

$$N \delta \epsilon = \epsilon \delta N$$

$$M \delta \varkappa = \varkappa \delta M \tag{31}$$

Consequently, equation (30) for the variation of the strain energy may be simplified to

$$\delta U = \iint\limits_{A} (N_x \delta \epsilon_x + N_\varphi \delta \epsilon_\varphi + N_{x\varphi} \delta \gamma_{x\varphi} - M_x \delta \varkappa_x - M_\varphi \delta \varkappa_\varphi + 2M_{x\varphi} \delta \varkappa_{x\varphi}) dA \tag{32}$$

Substitution of equations (11) and (12) into equation (32), with appropriate integration by parts and rearrangement of terms, yields

$$\frac{\delta_u U}{\sin \alpha} = -\iint\limits_{\varphi \, x} \left[x N_{x,x} + N_x - N_\varphi + \frac{N_{x\varphi,\varphi}}{\sin \alpha} \right] \delta u \, dx d\varphi + \int\limits_{\varphi} [x N_x \delta u]_{x_1}^{x_2} \, d\varphi$$

$$+ \int\limits_{x} \left[\frac{N_{x\varphi}}{\sin \alpha} \delta u \right]_{\varphi_1}^{\varphi_2} dx \tag{33}$$

$$\frac{\delta_v U}{\sin \alpha} = \iint\limits_{\varphi \, x} \left[\left(\frac{\overset{*}{M}_{\varphi,\varphi}}{x \sin \alpha} - \overset{*}{M}_{x\varphi,x} - \frac{2 \overset{*}{M}_{x\varphi}}{x} \right) \cot \alpha - \left(\frac{N_{\varphi,\varphi}}{\sin \alpha} + 2N_{x\varphi} + x N_{x\varphi,x} \right) \right.$$

$$\left. + \left(\frac{v N_\varphi \overset{\sim}{\cos} \alpha + w,_\varphi \overset{*}{N}_\varphi}{x \sin \alpha} + \overset{*}{N}_{x\varphi} w,_x \right) \cot \alpha \right] \delta v \, dx d\varphi$$

$$+ \int\limits_{\varphi} [(x N_{x\varphi} + M_{x\varphi} \overset{*}{\cot} \alpha) \delta v]_{x_1}^{x_2} \, d\varphi + \int\limits_{x} \left[\left(\frac{N_\varphi}{\sin \alpha} - \frac{\overset{*}{M}_\varphi \cot \alpha}{x \sin \alpha} \right) \delta v \right]_{\varphi_1}^{\varphi_2} dx$$

$$\tag{34}$$

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$$\frac{\delta_w \Gamma}{\sin \alpha} = -\int\int \left\{ \left[(xM_x),_{xx} + \frac{M_{\varphi,\varphi\varphi}}{x \sin^2 \alpha} - \frac{2M_{x\psi,x\varphi}}{\sin \alpha} - M_{\varphi,x} - \frac{2M_{x\varphi,\varphi}}{x \sin \alpha} \right] \right.$$

$$+ N_\varphi \cot \alpha + \frac{1}{\sin \alpha} [(w,_\varphi + v \overset{*}{\cos} \alpha)N_{x\varphi} + x \sin \alpha \, w,_x N_x],_x$$

$$\left. + \frac{1}{x \sin^2 \alpha} \left[(w,_\varphi + v \overset{*}{\cos} \alpha)N_\varphi + x \sin \alpha \, w,_x N_x\varphi \right],_\varphi \right\} \delta w \, dx d\varphi$$

$$+ \int_\varphi \left[\left\{ (xM_x),_x - M_\varphi - \frac{2M_{x\varphi,\varphi}}{\sin \alpha} + \frac{N_{x\varphi}}{\sin \alpha} (w,_\varphi + v \overset{*}{\cos} \alpha) + xw,_x N_x \right\} \delta w \right.$$

$$\left. - xM_x \, \delta w,_x \right|_{x_1}^{x_2} d\varphi$$

$$+ \int_x \left\{ \frac{1}{\sin \alpha} \left(\frac{M_{\varphi,\varphi}}{x \sin \alpha} - \frac{2M_{x\varphi}}{x} - 2M_{x\varphi,x} \right) + \frac{N_\varphi}{x \sin^2 \alpha}(w,_\varphi + v \overset{*}{\cos} \alpha) \right.$$

$$+ \frac{N_{x\varphi}w,_x}{\sin \alpha} \bigg\} \delta w - \frac{M_\varphi}{x \sin^2 \alpha} \, \delta w,_\varphi \bigg|_{\varphi_1}^{\varphi_2} + 2[(M_{x\varphi})_{\varphi_1}^{\varphi_2}]_{x_1}^{x_2} \tag{35}$$

EQUATIONS OF MOTION IN TERMS OF FORCES, MOMENTS AND DISPLACEMENTS

When equations (23), (33), (34), and (35) are, in turn, substituted into the variational formulation given by equations (26), three nonlinear partial differential equations of motion (Euler equations) are obtained for the conical shell in terms of forces, moments, and displacements:

$$N_{x,x} + \frac{N_x - N_\varphi}{x} + \frac{N_{x\varphi,\varphi}}{x \sin \alpha} = \rho h \ddot{u} \tag{36}$$

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$$\frac{N_{\varphi,\varphi}}{x \sin \alpha} + N_{x\varphi,x} + \frac{2N_{x\varphi}}{x} + \left(M^*_{x\varphi,x} + \frac{2M^*_{x\varphi}}{x} - \frac{M^*_{\varphi,\varphi}}{x \sin \alpha}\right)\frac{\cot \alpha}{x}$$

$$- \left[\left(\frac{w,^*_\varphi + v \overset{*}{\tilde{\cos}} \alpha}{x \sin \alpha}\right) N_\varphi + w,^*_x N_{x\varphi}\right]\frac{\cot \alpha}{x} = \rho h \ddot{v} \qquad (37)$$

$$\frac{(xM_x),_{xx}}{x} + \frac{M_{\varphi,\varphi\varphi}}{x^2 \sin^2 \alpha} - \frac{2M_{x\varphi,x\varphi}}{x \sin \alpha} - \frac{M_{\varphi,x}}{x} - \frac{2M_{x\varphi,\varphi}}{x^2 \sin \alpha}$$

$$+ \frac{1}{x \sin \alpha} \left\{ N_\varphi \cos \alpha + \left[(w,_\varphi + v \overset{*}{\cos} \alpha)N_{x\varphi} + x \sin \alpha\, w,_x N_x\right],_x \right.$$

$$\left. + \left[\left(\frac{w,_\varphi + v \overset{*}{\cos} \alpha}{x \sin \alpha}\right)N_\varphi + w,_x N_{x\varphi}\right],_\varphi \right\} = \rho h \ddot{w} \qquad (38)$$

EQUATIONS OF MOTION IN TERMS OF FORCES AND DISPLACEMENTS

Through the use of equations (11), (12), (17), and (18), the forces and moments may be obtained in terms of displacements. First, the forces and moments are written as functions of the displacements:

$$N_x = \frac{Eh}{1-\nu^2} \left\{ u,_x + \frac{1}{2} w,_x^2 + \nu\left[\frac{u}{x} + \frac{v,_\varphi - w \cos \alpha}{x \sin \alpha} + \frac{1}{2x^2 \sin^2 \alpha} \cdot \right.\right.$$

$$\left.\left. (w,_\varphi^2 + v^2 \overset{*}{\tilde{\cos}}^2 \alpha + 2vw,^*_\varphi \cos \alpha)\right]\right\}$$

$$N_\varphi = \frac{Eh}{1-\nu^2} \left\{ \frac{u}{x} + \frac{v,_\varphi - w \cos \alpha}{x \sin \alpha} + \frac{1}{2x^2 \sin^2 \alpha}(w,_\varphi^2 + v^2 \overset{*}{\tilde{\cos}}^2 \alpha \right.$$

$$\left. + 2vw,^*_\varphi \cos \alpha + \nu(u,_x + \frac{1}{2} w,_x^2)\right\} \qquad (39)$$

$$N_{x\varphi} = \frac{Eh}{2(1+\nu)} \left\{ v,_x + \frac{u,_\varphi}{x \sin \alpha} - \frac{v}{x} + \frac{w,_x}{x \sin \alpha}(w,_\varphi + v \overset{*}{\cos} \alpha)\right\}$$

and

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$$M_x = -D\left\{w,_{xx} + \left[\frac{w,_x}{x} + \frac{1}{x^2 \sin^2 \alpha}(w,_{\varphi\varphi} + v,_\varphi \cos \alpha)\right]\right\}$$

$$M_\varphi = -D\left\{\frac{w,_x}{x} + \frac{1}{x^2 \sin^2 \alpha}(w,_{\varphi\varphi} + v,_\varphi^* \cos \alpha) + \nu w,_{xx}\right\} \qquad (40)$$

$$M_{x\varphi} = D(1-\nu)\left\{\frac{w,_{x\varphi} + \frac{1}{2} v,_x^* \cos \alpha}{x \sin \alpha} - \frac{w,_\varphi + v \cos \alpha}{x^2 \sin \alpha}\right\}$$

Next, the three equations of motion given by equations (36)-(38) may be written in terms of forces and displacements by virtue of equations (40); that is,

$$N_{x,x} + \frac{N_x - N_\varphi}{x} + \frac{N_{x\varphi,\varphi}}{x \sin \alpha} = \rho h \ddot{u} \qquad (41)$$

$$\frac{N_{\varphi,\varphi}}{x \sin \alpha} + N_{x\varphi,x} + \frac{2N_{x\varphi}}{x} + \frac{D \cos \alpha}{x^2 \sin^2 \alpha}\left[w,_{xx\varphi}^* + \frac{w,_{\gamma\varphi}^*}{x} + \frac{w,_{\varphi\varphi\varphi}^* + v,_{\varphi\varphi}^* \cos \alpha}{x^2 \sin^2 \alpha}\right.$$

$$\left. + \frac{1-\nu}{2}\left(v,_{xx}^* - \frac{v,_x^*}{x}\right)\cos \alpha\right] - \frac{\cot \alpha}{x}\left[\left(\frac{w,_\varphi^* + v \cos \alpha}{x \sin \alpha}\right)N_\varphi\right.$$

$$\left. + w,_x N_{x\varphi}\right] = \rho h \ddot{v} \qquad (42)$$

$$D \nabla_c^4 w + \frac{D \cos \alpha}{x^2 \sin^2 \alpha}\left[(\nabla_c^2 v)^*,_\varphi - 4\left(\frac{v,_\varphi^*}{x}\right),_x\right] - \frac{1}{x \sin \alpha}\left\{N_\varphi \cos \alpha\right.$$

$$-[(w,_\varphi + v \cos \alpha)N_{x\varphi} + x \sin \alpha \, w,_x N_x],_x$$

$$\left. -\left[\left(\frac{w,_\varphi + v \cos \alpha}{x \sin \alpha}\right)N_\varphi + w,_x N_{x\varphi}\right],_\varphi\right\} = -\rho h \ddot{w} \qquad (43)$$

where the harmonic operator ∇_c^2 is defined as

$$\nabla_c^2(\ ) = (\ ),_{xx} + \frac{1}{x}(\ ),_x + \frac{1}{x^2 \sin^2 \alpha}(\ ),_{\varphi\varphi} \qquad (44)$$

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$$\nabla_c^4(\ ) = \nabla_c^2[\nabla_c^2(\ )] \tag{45}$$

EQUATIONS OF MOTION IN TERMS OF DISPLACEMENTS

Equations (41)-(43) may further be written in terms of displacements only by utilizing equations (39). The displacement equations of motion thus become

$$
\frac{E}{1-\nu^2}\Bigg\{\Bigg\{u_{,x} + \frac{1}{2}w_{,x} + \nu\Bigg[\frac{u}{x} + \frac{v_{,\varphi} - w\cos\alpha}{x\sin\alpha} + \frac{1}{2x^2\sin^2\alpha}(w_{,\varphi}^2 + v^2\overset{*}{\underset{\sim}{\cos}}{}^2\alpha
$$

$$
+ 2vw_{,\varphi}\overset{*}{\cos}\alpha\Bigg]\Bigg\}_{,x} + \frac{1-\nu}{x}\Bigg\{u_{,x} + \frac{1}{2}w_{,x}^2 - \frac{u}{x} - \frac{v_{,\varphi} - w\cos\alpha}{x\sin\alpha}
$$

$$
- \frac{1}{2x^2\sin^2\alpha}(w_{,\varphi}^2 + v^2\overset{*}{\underset{\sim}{\cos}}{}^2\alpha + 2vw_{,\varphi}\overset{*}{\cos}\alpha)\Bigg\}
$$

$$
+ \frac{1-\nu}{2x\sin\alpha}\Bigg\{v_{,x} + \frac{u_{,\varphi}}{x\sin\alpha} - \frac{v}{x} + \frac{w_{,x}}{x\sin\alpha}(w_{,\varphi} + v\overset{*}{\cos}\alpha\Bigg\}_{,\varphi}\Bigg\}
$$

$$
= \rho\ddot{u} \tag{46}
$$

$$
\frac{E}{1-\nu^2}\Bigg\{\frac{1}{x\sin\alpha}\Bigg[\frac{v_{,\varphi} - w\cos\alpha}{x\sin\alpha} + \frac{u}{x} + \frac{1}{2x^2\sin^2\alpha}(w_{,\varphi}^2 + v^2\overset{*}{\underset{\sim}{\cos}}{}^2\alpha
$$

$$
+ 2vw_{,\varphi}\overset{*}{\cos}\alpha) + \nu(u_{,x} + \frac{1}{2}w_{,x}^2)\Bigg]_{,\varphi} + \frac{1-\nu}{2}\Bigg[v_{,x} + \frac{u_{,\varphi}}{x\sin\alpha} - \frac{v}{x}
$$

$$
+ \frac{w_{,x}}{x\sin\alpha}(w_{,\varphi} + v\overset{*}{\cos}\alpha)\Bigg]_{,x} + \frac{1-\nu}{x}\Bigg[v_{,x} + \frac{u_{,\varphi}}{x\sin\alpha} - \frac{v}{x} + \frac{w_{,x}}{x\sin\alpha}\cdot
$$

$$
(w_{,\varphi} + v\overset{*}{\cos}\alpha)\Bigg] + \frac{h^2\cos\alpha}{12^2\sin^2\alpha}\Bigg[w_{,xx\varphi} + \frac{w_{,x\varphi}^{*}}{x} + \frac{w_{,\varphi\varphi\varphi}^{*} + v_{,\varphi\varphi}^{*}\cos\alpha}{x^2\sin^2\alpha}
$$

$$
+ \frac{1-\nu}{2}(v_{,xx} - \frac{v_{,x}}{x})\cos\alpha\Bigg] - \frac{\cot\alpha}{x}\Bigg\{\Bigg(\frac{w_{,\varphi} + v\overset{*}{\underset{\sim}{\cos}}\alpha}{x\sin\alpha}\Bigg)\Bigg[\frac{u}{v} + \frac{v_{,\varphi} - w\cos\alpha}{x\sin\alpha}
$$

(continued)

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$$+ \frac{1}{2x^2 \sin^2 \alpha}(w,_\varphi^2 + v^2 \overset{*}{\underset{\sim}{\cos}}^2 \alpha + 2vw,_\varphi^* \cos \alpha \cos \alpha) + \nu(u,_x + \frac{1}{2}w,_x^2) \Bigg]$$

$$+ \frac{w,_x(1-\nu)}{2} \left[v,_x + \frac{u,_\varphi}{x \sin \alpha} - \frac{v}{x} + \frac{w,_x}{x \sin \alpha}(w,_\varphi + v \overset{*}{\cos} \alpha) \right] \Bigg\} \Bigg\} = \rho \ddot{v}$$

$$(47)$$

$$D \nabla_c^4 w + \frac{D \cos \alpha}{x^2 \sin^2 \alpha} \left[(\nabla_c^2 v),_\varphi^* - 4 \left(\frac{v,_\varphi^*}{x} \right),_x \right] - \frac{Eh}{(1-\nu^2)x \sin \alpha} \Bigg[\cos \alpha \Bigg\{ \frac{u}{x}$$

$$+ \frac{v,_\varphi - w \cos \alpha}{x \sin \alpha} + \frac{1}{2x^2 \sin^2 \alpha}(w,_\varphi^2 + v^2 \overset{*}{\underset{\sim}{\cos}}^2 \alpha + 2vw,_\varphi^* \cos \alpha)$$

$$+ \nu(u,_x + \frac{1}{2}w,_x^2) \Bigg\} + \Bigg\{ (w,_\varphi + v \overset{*}{\cos} \alpha) \frac{1-\nu}{2} \left[v,_x + \frac{u,_\varphi}{x \sin \alpha} - \frac{v}{x} \right.$$

$$+ \frac{w,_x}{x \sin \alpha}(w,_\varphi + v \overset{*}{\cos} \alpha) \Bigg] + x \sin \alpha w,_x \Bigg\{ u,_x + \frac{1}{2}w,_x^2$$

$$+ \nu \left[\frac{v,_\varphi - w \cos \alpha}{x \sin \alpha} + \frac{u}{x} + \frac{1}{2x^2 \sin^2 \alpha}(w,_\varphi^2 + v^2 \overset{*}{\underset{\sim}{\cos}}^2 \alpha \right.$$

$$+ 2vw,_\varphi^* \cos \alpha) \Bigg] \Bigg\} \Bigg\},_x + \Bigg\{ \left(\frac{w,_\varphi + v \overset{*}{\cos} \alpha}{x \sin \alpha} \right) \left[\frac{v,_\varphi - w \cos \alpha}{x \sin \alpha} \right.$$

$$+ \frac{u}{x} + \frac{1}{2x^2 \sin^2 \alpha}(w,_\varphi^2 + v^2 \overset{*}{\underset{\sim}{\cos}}^2 \alpha + 2vw,_\varphi^* \cos \alpha)$$

$$+ \nu(u,_x + \frac{1}{2}w,_x^2) \Bigg] + \frac{w,_x(1-\nu)}{2} \left[v,_x + \frac{u,_\varphi}{x \sin \alpha} - \frac{v}{x} + \frac{w,_x}{x \sin \alpha}(w,_\varphi \right.$$

$$+ v \overset{*}{\cos} \alpha) \Bigg] \Bigg\},_\varphi \Bigg] = - \rho h \ddot{w} \quad (48)$$

BOUNDARY CONDITIONS

The associated boundary conditions which result from application of the variational principle [equations (25)] are given in terms of forces, moments, and displacements as follows:

(a) along $x = x_1$ and $x = x_2$

$$N_x = 0 \qquad\qquad \text{or } \delta u = 0$$

$$N_{x\varphi} + \frac{\overset{*}{M}_{x\varphi} \cos \alpha}{x \sin \alpha} = 0 \qquad\qquad \text{or } \delta v = 0$$

$$\sin \alpha (x M_x),_x - M_\varphi \sin \alpha - 2M_{x\varphi,\varphi} + x \sin \alpha \, w,_x N_x$$

$$+ N_{x\varphi}(w,_\varphi + v \overset{*}{\cos} \alpha) = 0 \qquad \text{or } \delta w = 0$$

$$M_x = 0 \qquad\qquad \text{or } \delta w,_x = 0$$

(49)

(b) along $\varphi = \varphi_1$ and $\varphi = \varphi_2$

$$N_{x\varphi} = 0 \qquad\qquad \text{or } \delta u = 0$$

$$N_\varphi - \frac{\overset{*}{M}_\varphi \cos \alpha}{x \sin \alpha} = 0 \qquad\qquad \text{or } \delta v = 0$$

$$M_{\varphi,\varphi} - 2 \sin \alpha (x M_{x\varphi}),_\varphi + x \sin \alpha \, w,_x N_{x\varphi}$$

$$+ N_\varphi(w,_\varphi + v \overset{*}{\cos} \alpha) = 0 \qquad\qquad \text{or } \delta w = 0$$

$$M_\varphi = 0 \qquad\qquad \text{or } \delta w,_\varphi = 0$$

(50)

(c) at the corners of the segment

$$M_{x\varphi} = 0 \qquad\qquad \text{or } \delta w = 0 \qquad (51)$$

These boundary conditions may be written also in terms of forces and displacements by using equations (40); thus,

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(a) along $x = x_1$ and $x = x_2$

$$N_x = 0 \qquad \underline{\text{or}} \quad \delta u = 0$$

$$N_{x\varphi} + \frac{D(1-\nu)\cos\alpha}{x^2 \sin^2\alpha}(w,_{x\varphi}^* + \frac{1}{2} v,_x^* \cos\alpha - \frac{w,_\varphi^* + v \cos\alpha}{x}) = 0$$

$$\underline{\text{or}} \quad \delta v = 0$$

$$-D\left\{x \sin\alpha\left\{w,_{xx} + \nu\left|\frac{w,_x}{x} + \frac{1}{x^2 \sin^2\alpha}(w,_{\varphi\varphi} + v,_\varphi^* \cos\alpha)\right|\right\}\right\},_x$$

$$+D \sin\alpha\left[\frac{w,_x}{x} + \frac{1}{x^2 \sin^2\alpha}(w,_{\varphi\varphi} + v,_\varphi^* \cos\alpha) + \nu w,_{xx}\right]$$

$$- \frac{2D(1-\nu)}{x \sin\alpha}\left|w,_{x\varphi} + \frac{1}{2} v,_x \cos\alpha - \frac{w,_\varphi^* + v \cos\alpha}{x}\right|,_\varphi$$

$$+ x \sin\alpha\, w,_x N_x + N_{x\varphi}(w,_\varphi + v \cos\alpha^*) = 0 \qquad \underline{\text{or}} \quad \delta w = 0$$

$$w,_{xx} + \nu\left[\frac{w,_x}{x} + \frac{1}{x^2 \sin^2\alpha}(w,_{\varphi\varphi} + v,_\varphi^* \cos\alpha)\right] = 0$$

$$\underline{\text{or}} \quad \delta w,_x = 0$$

$$\tag{52}$$

(b) along $\varphi = \varphi_1$ and $\varphi = \varphi_2$

$$N_{x\varphi} = 0 \qquad \underline{\text{or}} \quad \delta u = 0$$

$$N_\varphi + \frac{D \cot\alpha}{x}\left[\frac{w,_x^*}{x} + \frac{1}{x^2 \sin^2\alpha}(w,_{\varphi\varphi}^* + v,_\varphi^* \cos\alpha) + \nu w,_{xx}^*\right] = 0$$

$$\underline{\text{or}} \quad \delta v = 0$$

$$-D\left[\frac{w,_x}{x} + \frac{1}{x^2 \sin^2\alpha}(w,_{\varphi\varphi} + v,_\varphi^* \cos\alpha) + \nu w,_{xx}\right],_\varphi$$

$$- 2D(1-\nu)\left[w,_{x\varphi} + \frac{1}{2} v,_x \cos\alpha - \frac{w,_\varphi + v \cos\alpha^*}{x}\right],_x \quad \text{(continued)}$$

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$$+ x \sin \alpha \, w,_x N_{x\varphi} + N_\varphi (w,_\varphi + v \overset{*}{\cos} \alpha) = 0 \quad \underline{or} \quad \delta w = 0 \qquad (53)$$

$$\frac{w,_x}{x} + \frac{1}{x^2 \sin^2 \alpha} (w,_{\varphi\varphi} + v,_\varphi^* \cos \alpha) + \nu w,_{xx} = 0 \quad \underline{or} \quad \delta w,_\varphi = 0$$

(c) <u>at the corners</u>

$$w,_{x\varphi} + \frac{1}{2} v,_x^* \cos \alpha - (w,_\varphi + v \overset{*}{\cos} \alpha)/x \qquad \underline{or} \quad \delta w = 0 \qquad (54)$$

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REDUCTION TO CLASSICAL EQUATIONS

A simplified set of equations of motion may be obtained if simplified strain and curvature relations similar to the well-known von Kármán-Donnell relations for circular cylindrical shells are employed instead of the more accurate relations deduced in this study [equations (11) and (12)]. These simplified relations are the same as those used by Seide[9] and Singer[10] for the stability analysis of conical shells; they may be obtained by neglecting all terms marked by a $(^*)$ superscript in equations (11) and (12). Thus,

$$\left. \begin{aligned} \epsilon_x &= u,_x + \frac{1}{2} w,_x^2 \\[2mm] \epsilon_\varphi &= \frac{u}{x} + \frac{v,_\varphi - w \cos\alpha}{x \sin\alpha} + \frac{w,_\varphi^2}{2x^2 \sin^2\alpha} \\[2mm] \gamma_{x\varphi} &= v,_x + \frac{u,_\varphi}{x \sin\alpha} - \frac{v}{x} + \frac{w,_x w,_\varphi}{x \sin\alpha} \end{aligned} \right\} \qquad (55)$$

$$\left. \begin{aligned} \varkappa_x &= w,_{xx} \\[2mm] \varkappa_\varphi &= \frac{w,_{\varphi\varphi}}{x^2 \sin^2\alpha} + \frac{w,_x}{x} \\[2mm] \varkappa_{x\varphi} &= \frac{w,_{x\varphi}}{x \sin\alpha} - \frac{w,_\varphi}{x^2 \sin\alpha} \end{aligned} \right\} \qquad (56)$$

The force-displacement nonlinear equations of motion resulting from substitution of equations (55) and (56) into the Lagrangian and carrying out the variations in accordance with Hamilton's principle may be obtained here simply by neglecting the terms marked with a $(^*)$ superscript in equations (36)-(38); that is,

$$N_{x,x} + \frac{N_x - N_\varphi}{x} + \frac{N_{x\varphi,\varphi}}{x \sin\alpha} = \rho h \ddot{u}$$

$$\frac{N_{\varphi,\varphi}}{x \sin\alpha} + N_{x\varphi,x} + \frac{2N_{x\varphi}}{x} = \rho h \ddot{v} \qquad \text{(continued)}$$

$$D \, \nabla_c^4 w - \frac{1}{x \sin \alpha} \left\{ N_\varphi \cos \alpha + (w,_\varphi N_{x\varphi} + x \sin \alpha \, w,_x N_x),_x \right.$$

$$\left. + \left(\frac{w,_\varphi N_\varphi}{x \sin \alpha} + w,_x N_{x\varphi} \right),_\varphi \right\} = -\rho h \ddot{w} \tag{57}$$

If the motion of the shell walls during vibration is predominantly radial (as when there are a large number of circumferential waves in the modal pattern), then equations (57) admit yet a further simplification, namely, that of neglecting the midsurface inertia terms. The justification for this simplification is given by Reissner[13]. Thus, with the midsurface inertia terms omitted, the equations of motion become

$$\left. \begin{aligned} & N_{x,x} + \frac{N_x - N_\varphi}{x} + \frac{N_{x\varphi,\varphi}}{x \sin \alpha} = 0 \\[2mm] & \frac{N_{\varphi,\varphi}}{x \sin \alpha} + N_{x\varphi,x} + \frac{2N_{x\varphi}}{x} = 0 \\[2mm] & D \, \nabla_c^4 w - \frac{h}{x \sin \alpha} \left\{ N_\varphi \cos \alpha + (w,_\varphi N_{x\varphi} + x \sin \alpha \, w,_x N_x),_x \right. \\[1mm] & \left. \qquad + \left(\frac{w,_\varphi N_\varphi}{x \sin \alpha} + w,_x N_{x\varphi} \right),_\varphi \right\} = -\rho h \ddot{w} \end{aligned} \right\} \tag{58}$$

Using the first two equations of (58), the third may be rewritten as

$$D\nabla_c^4 w - \frac{1}{x \sin \alpha} \left[N_\varphi \left(\cos \alpha + \frac{w,_{\varphi\varphi}}{x \sin \alpha} + w,_x \sin \alpha \right) + N_x (x \sin \alpha \, w,_{xx}) \right.$$

$$\left. + 2x N_{x\varphi} \left(\frac{w,_\varphi}{x} \right),_x \right] = -\rho h \ddot{w} \tag{58a}$$

Equations (58) agree with those presented in Reference 25 for a nonlinear postbuckling analysis of a conical shell.

The three equations of motion given by equations (58) may be reduced in number by one through introduction of a stress function F defined by

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$$\frac{N_x}{h} = \frac{F,_x}{x} + \frac{F,_{\varphi\varphi}}{x^2 \sin^2 \alpha}$$

$$\frac{N_\varphi}{h} = F,_{xx} \qquad\qquad\qquad\qquad\qquad (59)$$

$$\frac{N_{x\varphi}}{h} = \frac{1}{x \sin \alpha}(F,_{x\varphi} - \frac{F,_\varphi}{x})$$

such that the first two equations of (58) are satisfied identically.

To establish a relationship between F and w independent of u and v, the strain compatibility equation for a conical shell, as employed, for example, by Seide[9],

$$\frac{\gamma_{x\varphi,\varphi}}{x^2 \sin \alpha} + \frac{\gamma_{x\varphi,x\varphi}}{x \sin \alpha} - \frac{\epsilon_{x,\varphi\varphi}}{x^2 \sin^2 \alpha} - \epsilon_{\varphi,xx} - \frac{2\epsilon_{\varphi,x}}{x} + \frac{\epsilon_{x,x}}{x} = \Gamma(w) \quad (60)$$

is used (where Γ is a function of w). Substitution of equations (55) into equation (60) gives

$$\Gamma(w) = \frac{w,_{xx} \cos \alpha}{x \sin \alpha} - \frac{1}{x^2 \sin^2 \alpha}\left[w,_{x\varphi} - w,_{xx}w,_{\varphi\varphi}\left(\frac{w,_\varphi^2}{x}\right) - x \sin^2 \alpha \, w,_x w,_{xx}\right]$$

$$(60a)$$

Substitution of the constitutive law [equation (17)] and equation (59) into equation (60), in conjunction with equation (60a), then yields the desired compatibility equation in terms of the stress function and radial displacement only as

$$\nabla_c^4 F = \frac{E}{x^2 \sin^2 \alpha}\left[w,_{x\varphi}^2 - w,_{xx}w,_{\varphi\varphi} - \left(\frac{w,_\varphi^2}{x}\right),_x - x \sin^2 \alpha \, w,_x w,_{xx}\right]$$

$$- \frac{Ew,_{xx} \cos \alpha}{x \sin \alpha} \qquad\qquad (61)$$

The lateral equilibrium equation may be written in terms of the stress function F and the radial displacement w by substitution of equations (59) into the third of equations (58); thus,

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$$D\nabla_c^4 w - \frac{h \cos \alpha}{x \sin \alpha} F,_{xx} - \frac{h}{x^2 \sin^2 \alpha} \left[w,_{xx} F,_{\varphi\varphi} - 2w,_{x\varphi} F,_{x\varphi} + w,_{\varphi\varphi} F,_{xx} \right.$$
$$\left. + 2\left(\frac{w,_\varphi F,_\varphi}{x}\right),_x + x \sin^2 \alpha (w,_x F,_x),_x \right] = -\rho h \ddot{w} \quad (62)$$

Equations (61) and (62) together represent the simplified nonlinear vibrational behavior of a conical shell. They are the conical shell counterparts of the von Kármán-Donnell equations for circular cylindrical shells. When equations (61) and (62) are linearized, the resulting equations are those describing the classical behavior of a conical shell in free vibration; thus,

$$D\nabla_c^4 w - \frac{h \cot \alpha}{x} F,_{xx} + \rho h \ddot{w} = 0 \quad (63)$$

and

$$\nabla_c^4 F = - \frac{E w,_{xx} \cot \alpha}{x} \quad (64)$$

These linear equations were solved by Seide in Reference 11.

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SUGGESTED METHODS OF SOLUTION

The differential equations (46), (47), and (48) reflect the use of the
complete strain and curvature relations given by equations (11) and (12);
they are three highly nonlinear coupled equations in the three dependent
variables u, v, and w. On the other hand, the simplified "classical"
strain and curvature relations given by equations (55), with the midsur-
face inertia terms neglected, admit reduction to two simultaneous
differential equations involving only the stress function F and the
radial displacement w. This reduced set of two simultaneous equations
[equations (61) and (62)] is still highly nonlinear and coupled but probably
more manageable in the classical sense than the initial set of three
given by equations (41)-(43). However, the classical equations are valid
only for motion of the shell in which a large number of circumferential
waves is present.

As a first step toward solving the conical shell equations described
above, it would be feasible to employ properly modified techniques used
previously for the limiting case of a circular cylindrical shell (as
discussed by Mayers and Wrenn[18]).

All of the existing analytical studies of the nonlinear vibrations of
thin circular cylindrical shells, except for those reported in References
18, 21, and 22, have the inherent assumptions (1) that the radial displace-
ment is representable by a chessboard pattern, (2) that the shell possesses
many circumferential waves (i.e., shallow-shell behavior), and (3) that
the midsurface inertia terms can be neglected. In accordance with
assumptions (2) and (3), all studies have involved the solution of the
classical equations [equations (103) of Appendix III] based upon von
Kármán-Donnell theory*. Assumption (1), however, seems to be intuitively
incorrect. As noted in References 18 and 21, use of the chessboard
pattern for the radial displacement leads to a violation of the periodicity

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*Mayers and Wrenn[18] also carried out a first-order approximation solution
of the equation set [equation (98)] based upon Sander's strain terms
with midsurface inertia terms retained.

condition for the circumferential displacement v given by

$$v(x,\varphi,t) = v(x,\varphi,2\pi,t) \tag{65}$$

or,

$$\int_0^{2\pi} v,_\varphi \, d\varphi = 0 \tag{66}$$

when finite displacements occur.

The fixed-parameter solution for a circular cylindrical shell, first presented by Evensen[22] on the basis of the Galerkin procedure and again by Mayers and Wrenn[18] as a special case using the Rayleigh-Ritz approach, may be extended to the conical shell case by assuming a radial displacement which satisfies both the geometric boundary conditions and the periodicity condition.

If the geometric boundary conditions are taken to be those for classical simple supports, namely

$$\left. \begin{array}{rcl} w(x,\varphi,t) & = w(x_2,\varphi,t) & = 0 \\[2mm] v(x,\varphi,t) & = v(x_2,\varphi,t) & = 0, \end{array} \right\} \tag{67}$$

then a corresponding second-order approximation for the radial displacement w which satisfies the boundary and periodicity conditions may be taken as

$$w = A_1(t) \, \sin \frac{m\pi(x-x_1)}{x_2-x_1} \cos n\varphi + \frac{n^2 A_1^2(t)}{4x\sin\alpha \, \cos\alpha} \, \sin^2 \frac{m\pi(x-x_1)}{x_2-x_1} \tag{68}$$

where $A_1(t)$ is an arbitrary function of time. However, by restricting the time-dependent coefficients to terms proportional to $A_1(t)$ and $A_1^2(t)$, the resultant motion is forced to be periodic, although not

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necessarily harmonic.

A Rayleigh-Ritz approach (as described in Reference 18 for a circular cylindrical shell) may then be used by substitution of the assumed deflected shape into the compatibility equation [equation (61)] and obtaining a particular solution for the stress function F. The stress function thus obtained and the assumed deflection function w should lead to satisfaction of the requirement that the circumferential displacement v be periodic in 2π. F and w are then substituted into the Lagrangian written in terms of stresses (and therefore in terms of the stress function) for midsurface behavior and in terms of w for bending behavior. The application of the variational principle relative to $A_1(t)$ would thus yield a second-order nonlinear differential equation, which through a single change of variable may be reduced to two coupled first-order nonlinear differential equations. These may then be solved by a standard Runga-Kutta numerical technique.

The solution obtained in the preceding manner will, as previously stated, be periodic. Mayers and Wrenn[18] waived this restriction for a circular cylindrical shell by allowing for a free-parameter solution. The boundary conditions were the same as those stipulated by Evensen[22] (free support). The boundary conditions will not significantly influence the frequencies of vibration when a large number of waves are present in the axial direction; thus, for long shells, on a minimum energy basis, the nonperiodic behavior governs.

Extending the approach to a conical shell, an assumed radial deflection shape may be taken as

$$w = A_1(t) \sin \frac{m\pi(x-x_1)}{x_2-x_1} \cos n\varphi + A_2(t) \cos \frac{2m\pi(x-x_1)}{x_2-x_1} + A_3(t) \qquad (69)$$

where $A_1(t)$, $A_2(t)$ and $A_3(t)$ are arbitrary functions of time. A displacement function of this type was also used in a study of the postbuckling behavior of conical shells presented in Reference 25.

Next, using the assumed displacement function w, the solution for the

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stress function F is obtained from the compatibility equation
[equation (61)], and the parameter $A_3(t)$ is then determined by impos-
ing the continuity condition on the circumferential displacement v. The
w and F functions thus obtained are finally substituted into the
Lagrangian (in a manner similar to that described previously for the
fixed-parameter case) and variations are taken with respect to $A_1(t)$
and $A_2(t)$. The resulting nonlinear differential equations may again
be solved using a Runga-Kutta numerical technique.

Application of this method to the circular cylindrical shell resulted
in a solution which displayed nonperiodic motion; the same type of
behavior can also be expected in the case of a long conical shell.

The two solution approaches described above involve much greater diffi-
culty when applied to a conical shell than when applied to a circular
cylinder, since (1) the classical equations of equilibrium [equations (61)
and (62)] for a conical shell are more complicated than those for a
circular cylindrical shell [equations (106) and (107) of Appendix III],
and (2) the stress-compatibility equation for the conical shell is a
biharmonic equation with variable coefficients. Nonetheless, the methods
of solution may still be used and they are presently undergoing further
study in connection with the overall research effort related to conical
shells.

When the number of circumferential waves is not large, the classical
equations are invalid and should not be used; rather, the equations of
motion [equations (41)-(43)] based on the general strains and curvatures
[equations (11) and (12)] with retained midsurface inertias should be
solved. Alternatively, a direct approach may be used by substituting
assumed time-dependent displacement functions directly into the total
potential energy as given by equation (23) and applying the variational
principle relative to the assumed time-dependent parameters.

Mayers and Wrenn[18] used the latter procedure for a circular cylindrical
shell; however, due to the obvious algebraic difficulties, only one
term in the radial displacement function was retained, thus eliminating

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the possibility of the shell deforming in its typically nonlinear fashion. As a result of this forced chessboard pattern, the resulting motion was periodic; however, for very small amplitudes of vib ation, the periodic solution is an excellent approximation.

When extending this approach to the case of a conical shell, displacement functions for u, v, and w which satisfy the geometric boundary conditions for w and the periodicity condition may be taken as

$$w = A_1(t) \sin \frac{m\pi(x-x_1)}{x_2-x_1} \cos n\varphi$$

$$u = A_2(t) \cos \frac{m\pi(x-x_1)}{x_2-x_1} \cos n\varphi + A_3(t) \cos \frac{2m\pi(x-x_1)}{x_2-x_1} \cos 2n\varphi$$

$$+ A_4(t) \cos \frac{2m\pi(x-x_1)}{x_2-x_1} \qquad (70)$$

$$v = A_5(t) \sin \frac{m\pi(x-x_1)}{x_2-x_1} \sin n\varphi + A_6(t) \sin \frac{2m\pi(x-x_1)}{x_2-x_1} \sin 2n\varphi$$

$$+ A_7(t) \sin 2n\varphi$$

Substitution of equations (70) into equations (23) and (24) and the enforcement of the simultaneous vanishing of the first variation of the Lagrangian with respect to the various $A_i(t)$ functions would lead to a set of seven second-order coupled nonlinear differential equations. After a change of the independent variable and subsequent reduction of the seven equations to fourteen first-order nonlinear differential equations, the solution is then obtained using a standard Kutta-Merson numerical technique.

The solution techniques described above all suffer from a common disadvantage; namely, the number of terms which may be retained in the assumed displacement functions are limited as a result of algebraic complications. However, a numerical approach is presently being studied in which a general series solution for the various displacements may be assumed and a large number of terms retained. The method may best be

understood by demonstrating it for the problem of the nonlinear vibration of a beam. This is accomplished in Appendix V.

The proposed method basically involves the selection of assumed displacement functions in the general series form

$$w = \sum_{i=1}^{\ell} \Phi_i(x,\varphi) \, A_i(t)$$

$$v = \sum_{i=\ell+1}^{m} \Phi_i(x,\varphi) \, A_i(t) \tag{71}$$

$$u = \sum_{i=m+1}^{n} \Phi_i(x,\varphi) \, A_i(t)$$

where $A_i(t)$, $i=1,2,\ldots n$ are arbitrary functions of time.

Substitution of equations (71) into equations (23) and (24) and the enforcement of the simultaneous vanishing of the first variation of the Lagrangian with respect to the various $A_i(t)$ (<u>without multiplying out the series or carrying out the integrations</u>) will lead to a set of n second-order coupled nonlinear differential equations. After a change of the independent variable and subsequent reduction of the n second-order differential equations, the solutions may be obtained using the numerical procedure demonstrated in Appendix V.

The most general set of appropriate displacements for a conical shell which satisfies the simply supported conditions and periodicity of the circumferential displacement would be equations (71) with

$$\Phi_i(x,\varphi) = \sin i\beta(x-x_1) \sum_{j=1}^{\ell'} \cos j\varphi \qquad , \qquad 1 \leq i \leq \ell$$

$$\Phi_i(x,\varphi) = \cos(i-\ell)\beta(x-x_1) \sum_{j=\ell'+1}^{m'} \cos j\varphi \qquad , \qquad \ell+1 \leq i \leq m \tag{72}$$

$$\Phi_i(x,\varphi) = \sin(i-m)\beta(x-x_1) \sum_{j=m'+1}^{n'} \sin j\varphi \qquad , \qquad m+1 \leq i \leq n$$

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where

$$\beta = \frac{x}{x_2 - x_1} \tag{73}$$

In essence, $\Phi_i(x, t)$ may be taken as any type of function; the numerical technique does not differentiate between types of functions and is therefore not limited in this respect.

The main merit of this proposed numerical technique is that there is no necessity of expanding, multiplying, and integrating the assumed displacement functions analytically at any time during the solution. Further merits and some drawbacks of the method are discussed in Appendix V.

Finally, another method of solution, which involves the use of a modified-Reissner variational principle, should be investigated. This method was introduced in Reference 28 and again used successfully in Reference 29. It involves writing the strain energy of the system in such a manner, that Hooke's law [equations (17)] is not necessarily enforced prior to variation for middle-surface stresses and strains though it is forced for the bending terms. This approach is shown to be highly convergent and accurate for both thin plate and shell problems undergoing finite deflections even in the presence of inelastic deformations.

This method of solution and all the others previously described are presently being investigated. The results and conclusions are expected to provide the same level of knowledge for the nonlinear vibrations and maximum strength of conical shells as has now been developed for flat plates and circular cylindrical shells by Mayers et al.[28,29,32,33].

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CONCLUSIONS

The governing differential equations and associated boundary conditions
for the nonlinear vibrations of a thin, truncated conical shell have been
developed by means of Hamilton's principle. The equations are based upon
the nonlinear strain-displacement relations deduced from those developed
by Sanders for thin shells of arbitrary shape.

Governing differential equations and boundary conditions have also been
developed for the nonlinear behavior of a conical shell in the presence
of static and dynamic loads. These equations apply equally well to
forced nonlinear vibration problems, nonlinear vibrations of a prestressed
shell and nonlinear prebuckling and postbuckling behavior of an arbitrarily
loaded shell. Linearized stability equations and boundary conditions
using the nonlinear terms of this study have also been developed.

All equations derived in this report have been shown to reduce (by a set
of simple substitutions) to corresponding equations for circular cylin-
drical shells and circular flat plates. The equations have also been
shown to reduce to the so-called "classical" conical shell equations
when the circumferential displacement terms in the curvature and nonlinear
strain relations are neglected and the midsurface inertias are omitted.

Various methods of solution extending from those applied in similar
investigations of circular cylindrical shells have been presented, and
a modified-Reissner variational principle method of solution has been
suggested. Finally, a new direct numerical solution scheme has been
proposed and demonstrated analytically.

Future efforts in this area should include implementation of the proposed
methods of solution for the nonlinear vibrations of a conical shell and
a comparison of the various methods. The modified-Reissner variational
principle approach and the new numerical scheme should at first be
employed to obtain solutions for the nonlinear vibrations of a circular
cylindrical shell. Comparison with the previous solutions should
determine the merits (if any) of these methods, and it may then be
decided whether they should be applied to the conical shell problem.

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Finally, these same methods of solution should be applied to nonlinear forced vibration problems and to the important design problems of the nonlinear postbuckling behavior and maximum strength of initially imperfect conical shells.

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EQUATIONS FOR THE NONLINEAR BEHAVIOR OF CONICAL SHELLS
UNDER ARBITRARY LOADS

Equations for the nonlinear behavior of conical shells acted upon by externally applied arbitrary loads may be derived without significant further difficulty by adding to the strain energy U [equation (14)] the potential of the applied loads; that is,

$$U_T = U + V_L + V_m + V_b \tag{74}$$

where U_T is the total potential energy of the shell in the presence of applied loads. V_L is the potential of the applied surface loads $(q_x, q_\varphi,$ and $q_z)$, V_m is the potential of the applied edge loads $(\bar{N}_x, \bar{N}_\varphi,$ and $\bar{N}_{x\varphi})$, and V_b is the potential of the applied edge moments and shear forces $(\bar{M}_x, \bar{M}_\varphi, \bar{M}_{x\varphi}, \bar{Q}_x,$ and $\bar{Q}_\varphi)$.

The applied-load potentials may be represented as

$$V_L = -\iint_{\varphi\ x} (q_x u + q_\varphi v + q_z w)\ x\ \sin\alpha\ dx d\varphi \tag{75}$$

$$V_m = -\int_\varphi \left[(\bar{N}_x u + \bar{N}_{x\varphi} v)\ x\ \sin\alpha \right]_{x_1}^{x_2} d\varphi\ -\int_x \left[\bar{N}_\varphi v + \bar{N}_{x\varphi} u \right]_{\varphi_1}^{\varphi_2} dx \tag{76}$$

$$V_b = -\int_\varphi \left[x \sin\alpha\ (-\bar{M}_x w,_x + \frac{\bar{M}_{x\varphi} w,_\varphi}{x \sin\alpha} + \bar{Q}_x w) \right]_{x_1}^{x_2} d\varphi$$

$$-\int_x \left[-\frac{\bar{M}_\varphi w,_\varphi}{x \sin\alpha} + \bar{M}_{x\varphi} w,_x + \bar{Q}_\varphi w \right]_{\varphi_1}^{\varphi_2} dx \tag{77}$$

The required variations are

$$\delta_u V_L = -\int_\varphi \int_x q_x \, x \, \sin\alpha \, \delta u \, dx d\varphi$$

$$\delta_v V_L = -\int_\upsilon \int_x q_\varphi x \, \sin\alpha \, \delta v \, dx d\varphi \qquad \qquad (78)$$

$$\delta_w V_L = -\int_\varphi \int_x q_z x \, \sin\alpha \, \delta w \, dx d\varphi$$

$$\delta_u V_m = -\sin\alpha \int_\varphi [x\bar{N}_x \, \delta u]_{x_1}^{x_2} \, d\varphi + \int_x [\bar{N}_{x\varphi}\delta u]_{\varphi_1}^{\varphi_2} \, dx$$

$$\delta_v V_m = -\sin\alpha \int_\varphi [x\bar{N}_{x\varphi} \, \delta v]_{x_1}^{x_2} \, d\varphi + \int_x [\bar{N}_\varphi \, \delta v]_{\varphi_1}^{\varphi_2} \, dx \qquad (79)$$

$$\delta_w V_m = 0$$

$$\delta_u V_b = 0$$

$$\delta_v V_b = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (80)$$

$$\delta_w V_b = \sin\alpha \int_\varphi \left[x \left(\frac{\bar{M}_{x\varphi,\varphi}}{x \sin\alpha} - \bar{Q}_x \right) \delta w \right]_{x_1}^{x_2} d\varphi + \int_x [(\bar{M}_{x\varphi,x} - \bar{Q}_\varphi) \, \delta w]_{\varphi_1}^{\varphi_2} dx$$

$$+ \sin\alpha \int_\varphi \left[x\bar{M}_x \, \delta w,_x \right]_{x_1}^{x_2} d\varphi + \int_x \left[\frac{\bar{M}_\varphi}{x \sin\alpha} \, \delta w,_\varphi \right]_{\varphi_1}^{\varphi_2} dx - 2[[\bar{M}_{x\varphi}\delta w]_{\varphi_1}^{\varphi_2}{}_{x_1}^{x_2}]$$

Equations (78) through (80) are added into the variational equations
[equations (26)] such that

$$\delta_{u,v,w} \int_t (T - U_\Gamma) \, dt = \delta_{u,v,w} \int_t (T - U - V_L - V_m - V_b) \, dt = 0 \qquad (81)$$

where the expressions for $\delta_{u,v,w}T$ and $\delta_{u,v,w}U$ are given by equations
(27) and (33)-(35), respectively.

The variation results in three Euler equations, which are the same as
those given by equations (36)-(38) with the addition of the surface loads

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q_x, q_φ and q_z, respectively, to the left-hand side of each equation. The equations of motion in terms of stresses and displacements are the same as those given by equations (41)-(43) with the respective surface loads added to the left-hand sides of the first two equations and to the right-hand side of the third; the same is true for the displacement equations of motion given by equations (46)-(48).

The resulting boundary conditions are

(a) along $x = x_1$ and $x = x_2$ (82)

$$N_x = \bar{N}_x \qquad\qquad \underline{or}\quad \delta u = 0$$

$$N_{x\varphi} - \frac{M_{x\varphi}^* \cos \alpha}{x \sin \alpha} = \bar{N}_{x\varphi} \qquad\qquad \underline{or}\quad \delta v = 0$$

$$(xM_x)_{,x} - M_\varphi - \frac{2M_{x\varphi,\varphi}}{\sin \alpha} + \frac{N_{x\varphi}}{\sin \alpha}(w_{,\varphi} + v \cos \alpha) + xw_{,x}N_x^*$$

$$= x\bar{Q}_x - \frac{\bar{M}_{x\varphi,\varphi}}{\sin \alpha} \qquad\qquad \underline{or}\quad \delta w = 0$$

$$M_x = \bar{M}_x \qquad\qquad \underline{or}\quad \delta w_{,x} = 0$$

(b) along $\varphi = \varphi_1$ and $\varphi = \varphi_2$ (83)

$$N_{x\varphi} = \bar{N}_{x\varphi} \qquad\qquad \underline{or}\quad \delta u = 0$$

$$N_\varphi - \frac{M_\varphi^* \cos \alpha}{x \sin \alpha} = \bar{N}_\varphi \qquad\qquad \underline{or}\quad \delta v = 0$$

$$\frac{M_{\varphi,\varphi}}{x \sin \alpha} - \frac{2(xM_{x\varphi})_{,x}}{x} + \frac{N_\varphi}{x \sin \alpha}(w_{,\varphi} + v \cos \alpha)^*$$

$$+ N_{x\varphi}w_{,x} = \bar{Q}_\varphi - \bar{M}_{x\varphi,x} \qquad\qquad \underline{or}\quad \delta w = 0$$

$$M_\varphi = \bar{M}_\varphi \qquad\qquad \underline{or}\quad \delta w_{,\varphi} = 0$$

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(c) <u>at the corners of the segment</u>   (84)

$$M_{x\varphi} = \bar{M}_{x\varphi} \qquad\qquad\qquad \underline{or} \quad \delta w = 0$$

All equations and boundary conditions presented in this appendix reduce to those of classical conical shell theory[9,25] by linearization of the equations and neglect of those terms marked by a (*) superscript.

APPENDIX II

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS FOR THE
STABILITY OF CONICAL SHELLS

Additional definition of symbols:

M_x, M_φ, $M_{x\varphi}$ additional moments caused by buckling

M_{x_o}, M_{φ_o}, $M_{x\varphi_o}$ moments prior to buckling

N_x, N_φ, $N_{x\varphi}$ additional forces caused by buckling

N_{x_o}, N_{φ_o}, $N_{x\varphi_o}$ forces prior to buckling

ϵ_x, ϵ_φ, $\gamma_{x\varphi}$ linear strain terms

ϵ'_x, ϵ'_φ, $\gamma'_{x\varphi}$ nonlinear strain terms

Linearized stability equations and boundary conditions for a conical shell may be readily obtained from the general equations of nonlinear behavior of conical shells derived in Appendix I.

The equilibrium at the time of buckling is obtained by consideration of the additional work done during buckling. The displacements u, v, and w are now the additional displacements caused by buckling, and the pre-buckling displacements are assumed to be small so that the additional displacements can be related to the undeformed geometry of the shell. The additional strains are taken as the nonlinear terms of equations (11), namely*,

$$\epsilon'_x = \frac{1}{2} w,_x^2$$

$$\epsilon'_\varphi = \frac{1}{2x^2 \sin^2 \alpha} (w,_\varphi^2 + v^2 \overset{*}{\cos^2} \alpha + 2vw,_\varphi \overset{*}{\cos} \alpha)$$

---

*These nonlinear terms differ from those used in previous studies[9,10,25] by retention of the tangential displacements terms.

$$\dot{\ }_{x\tau} = \frac{u_{,x}}{x \sin \alpha} (v_{,\tau} + v \cos \alpha) \tag{85}$$

The internal forces and moments N and M of equations (33)-(35), with the nonlinear terms omitted, must now be replaced by $N + N_0$ and $M + M_0$ respectively, and the potential of the membrane forces prior to buckling $(N_{x_0}, N_{\tau_0}, \text{ and } N_{x\tau_0})$ must be added to the variational equations [equations (81)] with the kinetic energy term omitted. Basically, with A representing the surface area, this means that,

$$\delta \tilde{U}_B = \iint_A [(N_x + N_{x_0}) \delta \epsilon_{x_0} + (N_\tau + N_{\tau_0}) \delta \epsilon_{\tau_0} + (N_{x\tau} + N_{x\tau_0}) \delta \gamma_{x\tau_0}] dA$$

$$- \iint_A [(M_x + M_{x_0}) \delta \kappa_x + (M_\tau + M_{\tau_0}) \delta \kappa_\tau - 2(M_{x\tau} + M_{x\tau_0}) \delta \kappa_{x\tau}] dA$$

$$+ \delta U_m + \delta U_L + \delta U_b + \delta U_m = 0 \tag{86}$$

where U_B is the total potential energy during buckling and $\bar{U}_m$ is the strain energy stored during buckling by the middle-surface forces of the unbuckled state; that is

$$\delta \bar{U}_m = \iint_A [N_{x_0} \delta \epsilon_x + N_{\tau_0} \delta \epsilon_\tau + N_{x\tau_0} \delta \gamma_{x\tau}] dA \tag{87}$$

The expressions for δU_L, δU_b, and δU_m are given by equations (75) through (80).

After equations (36)-(38) with the nonlinear terms omitted (and surface load terms added as per Appendix I) are also considered, the stability equations and boundary conditions are obtained as

$$N_{x,x} + \frac{N_x - N_\tau}{x} + \frac{N_{x\tau,\tau}}{x \sin \alpha} = 0 \tag{88}$$

$$\frac{N_{\tau,\tau}}{x \sin \alpha} + N_{x\tau,x} + \frac{2N_{x\tau}}{x} + \frac{\cot \alpha}{x} \left(M^*_{x\tau,x} + \frac{2M^*_{x\tau}}{x} - \frac{M^*_{\tau,\tau}}{x \sin \alpha} \right)$$

$$- \frac{\cot \alpha}{x} \left[\left(\frac{v_{,\tau}^* + v \, \widetilde{\cos} \, \alpha}{x \sin \alpha} \right) N_{\tau_0} + v_{,x} N_{x\tau_0} \right] = 0 \tag{89}$$

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$$(xM_x)_{,xx} + \frac{M_{\varphi,\varphi}}{x^2 \sin^2 \alpha} - \frac{2M_{x\varphi,x\varphi}}{x \sin \alpha} - \frac{M_{\varphi,x}}{x} - \frac{2M_{x\varphi,\varphi}}{x^2 \sin \alpha}$$

$$+ \frac{\cot \alpha}{x} M_\varphi + \frac{1}{x \sin \alpha} \left\{ \left| (w_{,\varphi} + v \cos \alpha) \overset{*}{N}_{x\varphi_o} \right. \right.$$

$$+ x \sin \alpha \, w_{,x} N_{x_o} \left|_{,x} + \left\{ \left(\frac{w_{,\varphi} + v \cos \alpha}{x \sin \alpha} \right) \overset{*}{N}_{\varphi_o} \right. \right.$$

$$\left. \left. + w_{,x} \overset{*}{N}_{x\varphi_o} \right|_{,\varphi} \right\} = 0 \tag{90}$$

The boundary conditions are*

(a) along $x=x_1$ and $x=x_2$

$$N_x = 0 \qquad\qquad \underline{or} \quad \delta u = 0$$

$$N_{x\varphi} + \frac{\overset{*}{N}_{x\varphi} \cos \alpha}{x \sin \alpha} = 0 \qquad\qquad \underline{or} \quad \delta v = 0$$

$$\sin \alpha \, (xM_x)_{,x} - M_\varphi \sin \alpha - 2M_{x\varphi,\varphi} \tag{91}$$

$$+ x \sin \alpha \, w_{,x} \overset{*}{N}_{x_o} + (w_{,\varphi} + v \cos \alpha) \overset{*}{N}_{x\varphi_o} = 0 \qquad \underline{or} \quad \delta w = 0$$

$$M_x = 0 \qquad\qquad \underline{or} \quad \delta w_{,x} = 0$$

(b) along $\varphi = \varphi_1$ and $\varphi = \varphi_2$

$$N_{x\varphi} = 0 \qquad\qquad \underline{or} \quad \delta u = 0$$

$$N_\varphi - \frac{\overset{*}{N}_\varphi \cos \alpha}{x \sin \alpha} = 0 \qquad\qquad \underline{or} \quad \delta v = 0$$

$$M_{\varphi,\varphi} - 2 \sin \alpha \, (xM_{x\varphi})_{,x} \tag{92}$$

$$+ (w_{,\varphi} + v \cos \alpha) \overset{*}{N}_{\varphi_o} + x \sin \alpha \, w_{,x} \overset{*}{N}_{x\varphi_o} = 0 \qquad \underline{or} \quad \delta w = 0$$

(continued)

---

*Equations (49)-(51), with the nonlinear terms omitted, must also
be used in the derivation.

$$M_{\varphi} = 0 \qquad\qquad \underline{\text{or}}\quad \delta w,_{\varphi} = 0$$

(c) <u>at the corners of the segment</u>

$$M_{x\varphi} = 0 \qquad\qquad \underline{\text{or}}\quad \delta w = 0 \qquad (93)$$

The stability equations and boundary conditions presented above may also be written in terms of either forces and displacements or displacements only by using equations (39) and (40). In this study, however, these rewritten equations may be obtained directly from equations (41)-(43), (46)-(48), and (52)-(54) simply by deleting the inertia terms and applying a zero subscript to the forces in the nonlinear terms consisting of products of forces and displacements.

The stability equations and boundary conditions derived in this appendix represent linear buckling theory based upon the strains and curvatures defined by equations (11) and (12) respectively. All the foregoing equations reduce to those presented by Seide[9], Singer[19], and Schnell[25] when terms marked by a (*) superscript are deleted.

REDUCTION OF THE GENERAL NONLINEAR CONICAL SHELL EQUATIONS TO THOSE FOR A CIRCULAR CYLINDRICAL SHELL

In the present study, general nonlinear equations for the bending,
buckling, and vibrations of conical shells have been developed. The
geometry of a truncated conical shell is such that it develops into a
circular cylindrical shell of radius $R = x \sin \alpha$ as the semi-vertex
angle α approaches zero. Accordingly, all equations developed in the
present study for a conical shell reduce to those for a circular cylin-
drical shell (as presented by Mayers and Wrenn[18] based on Sanders'
theory) through the simple set of substitutions

$$x \sin \alpha = R$$

$$\alpha = 0$$

$$\frac{1}{x} = 0 \qquad\qquad (94)$$

$$\varphi = \frac{y}{R}$$

where y is the tangential coordinate on the shell middle surface in the
circumferential direction. Some of the more important relationships
derived through use of the above substitution are listed in this
appendix for completeness.

MIDSURFACE STRAIN-DISPLACEMENT RELATIONS (rotation about the normal
neglected).

By virtue of equations (94), equations (11) reduce to

$$\epsilon_x = u,_x + \frac{1}{2} w,_x^2$$

$$\epsilon_y = v,_y - w/R + \frac{1}{2}(w,_y^2 + v^2/R^2 + 2vw,_y^*/R) \qquad (95)$$

$$\gamma_{xy} = v,_x + u,_y + w,_x(w,_y^* + v/R)$$

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CURVATURE-DISPLACEMENT RELATIONS

Equations (12) reduce to

$$\varkappa_x = w,_{xx}$$

$$\varkappa_\varphi = w,_{yy} + v,_y^{*}/R \tag{96}$$

$$\varkappa_{x\varphi} = w,_{xy} + \frac{1}{2} v,_x^{*}/R$$

When the terms marked by a superscript $(\overset{*}{\,})$ are deleted, the equations reduce to those of classical von Kármán-Donnell theory.

TOTAL POTENTIAL ENERGY

The total potential energy is given by

$$U_T = U_m + U_b + V_L + V_m + V_b \tag{97}$$

Reduction of equations (23) and (75) through (77), in conjunction with equation (97), yields

$$
\begin{aligned}
U_T = -\frac{Eh}{2(1-\nu^2)} \iint_{y\,x} &\left\{ (u,_x + \frac{1}{2} w,_x^2)^2 + [v,_y - w/R + \frac{1}{2}(w,_y^2 + \overset{\sim}{v^2}\!/R^2 \overset{*}{}\right. \\
&+ 2vw,_y^{*}/R)]^2 + 2\nu(u,_x + \frac{1}{2} w,_x^2)\, [v,_y - w/R + \frac{1}{2}(w,_y^2 \\
&+ \overset{\sim}{v^2}\!/R^2 \overset{*}{} + 2vw,_y^{*}/R)] + \frac{1-\nu}{2} [v,_x + u,_y + w,_x(w,_y \\
&+ v/R \overset{*}{})]^2 \Big\} dxdy + \frac{D}{2}\iint_{y\,x}\Big\{ w,_{xx}^2 + (w,_{yy} + v,_y^{*}/R)^2 + 2vw,_{xx}(w,_{yy}^{*}+v,_y^{*}/R) \\
&+ 2(1-\nu)(w,_{xy} + v,_x^{*}/2R^2 \Big\} \; dxdy \; -\iint_{y\,x} (q_x u + q_y v + q_z w)\, dxdy \\
&-\int_y [-\bar{N}_x u + \bar{N}_{xy} v]_{x_1}^{x_2} dy - \int_x [\bar{N}_y v + \bar{N}_{xy} u]_{y_1}^{y_2} dx - \int_y [-\bar{M}_x w,_x \\
&+ \bar{M}_{xy} w,_y + \bar{Q}_x w]_{x_1}^{x_2} dy - \int_x [-\bar{M}_y w,_y + \bar{M}_{xy} w,_x + \bar{Q}_y w]_{y_1}^{y_2} dx \tag{98}
\end{aligned}
$$

The expression given by equation (98), with the applied load terms deleted, was used by Mayers and Wrenn[18] in obtaining a first-approximation solution to the free vibration behavior of a circular cylindrical shell undergoing large displacements and moderate rotations.

FORCE-DISPLACEMENT EQUATIONS OF MOTION

Reduction of equations (41)-(43), with the addition of the applied load terms, yields

$$N_{x,x} + N_{xy,y} = \rho h \ddot{u} - q_x$$

$$N_{y,y} + N_{xy,x} + (D/R^2)[R(w,^*_{xxy} + w,^*_{yyy}) + v,^*_{yy} + \frac{1}{2}(1-\nu)v,^*_{xx}]$$

$$- [(w,^*_y + \tilde{v}/R)N_y + w,_x N_{xy}]/R = \rho h \ddot{v} - q_y$$

$$D\nabla^4 w + D(\nabla^2 v),_y/R - N_y/R - [(w,^*_y + v/R)N_{xy} + w,_x N_x],_x$$

$$- [(w,^*_y + v/R)N_y + w,_x N_{xy}],_y = q_z - \rho h \ddot{w}$$

(99)

BOUNDARY CONDITIONS

Equation (49)-(51) reduce to

(a) <u>along $x = x_1$ and $x = x_2$</u>

$$N_x = \overline{N}_x \qquad \qquad \underline{or} \quad \delta u = 0$$

$$N_{xy} + M,^*_{xy}/R = \overline{N}_{xy} \qquad \qquad \underline{or} \quad \delta v = 0$$

$$M_{x,x} - 2M_{xy,y} + N_{xy}(w,^*_y + v/R) \qquad \qquad (100)$$

$$+ w,_x N_x = \overline{Q}_x - \overline{M}_{xy,y} \qquad \qquad \underline{or} \quad \delta w = 0$$

$$M_x = \bar{M}_x \qquad \underline{\text{or}} \quad \delta w,_x = 0$$

(b) <u>along $y = y_1$ and $y = y_2$</u> (101)

$$N_{xy} = \bar{N}_{xy} \qquad \underline{\text{or}} \quad \delta u = 0$$

$$N_y - M_y^*/R = \bar{N}_y \qquad \underline{\text{or}} \quad \delta v = 0$$

$$M_{y,y} - 2M_{xy,x} + N_y^*(w,_y + v/R)$$

$$+ w,_x N_{xy} = \bar{Q}_y - \bar{M}_{xy,x} \qquad \underline{\text{or}} \quad \delta w = 0$$

$$M_y = \bar{M}_y \qquad \underline{\text{or}} \quad \delta w,_y = 0$$

(c) <u>at the corners of the segment</u> (102)

$$M_{xy} = \bar{M}_{xy} \qquad \underline{\text{or}} \quad \delta w = 0$$

CLASSICAL NONLINEAR EQUATIONS

Reduction of equations (58) yields the classical nonlinear differential
equations of motion for a circular cylindrical shell having a large
number of circumferential waves[18,22]; that is,

$$N_{x,x} + N_{xy,y} = 0$$

$$N_{y,y} + N_{xy,x} = 0 \tag{103}$$

$$D\nabla^4 w - N_y/R - (w,_x N_x + w,_y N_{xy}),_x - (w,_x N_{xy} + w,_y N_y) = -\rho h \ddot{w}$$

Using the first two of equations (103), the third can be rewritten as

$$D\nabla^4 w - N_y/R - (N_x w,_{xx} + N_y w,_{yy} + 2N_{xy} w,_{xy}) = -\rho h \ddot{w} \tag{104}$$

The stress function definition of equations (59) reduces to

$$N_x = hF,_{yy} \qquad \qquad \text{(continued)}$$

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$$N_y = hF,_{xx}$$

$$N_{xy} = -hF,_{xy} \tag{105}$$

The compatibility relationship [equation (61)] reduces to

$$\nabla^4 F = E(w,_{xy}^2 - w,_{xx}w,_{yy}) - Ew,_{xx}/R \tag{106}$$

and the lateral equilibrium equation [equation (62)] becomes

$$D\nabla^4 w - hF,_{xx}/R - h(w,_{xx}F,_{yy} - 2w,_{xy}F,_{xy} + w,_{yy}F,_{xx}) = -\rho h\ddot{w} \tag{107}$$

Equations (106) and (107) represent accurately only the free nonlinear vibrational behavior of a circular cylindrical shell having a large number of circumferential waves. Various approximate solutions for periodic behavior have recently been obtained for these equations, notably those by Chu[19], Nowinski[20], and Evensen[22]. The most recent solution, based on the energy formulation of the problem, is that obtained by Mayers and Wrenn[18]. In their work, previous periodic solutions are discussed and their validity is questioned on a minimum-energy-criterion basis.

Equations (106) and (107) with the radial inertia term omitted are one of the equation sets commonly referred to as the von Kármán-Donnell equations governing the large deflection behavior of thin plates and shells in the presence of midsurface forces.

All of the conical shell equations derived in Appendix I for nonlinear behavior under arbitrary loads and in Appendix II for stability behavior may also be easily reduced to corresponding equations for circular cylindrical shells through the substitutions given by equations (94).

APPENDIX IV

REDUCTION OF THE GENERAL NONLINEAR CONICAL SHELL EQUATIONS TO THOSE
FOR A CIRCULAR FLAT PLATE

In Appendix III the limiting case of a circular cylindrical shell is considered and the basic equations describing its nonlinear behavior are easily derived from the conical shell equations. Another limiting case occurs when the semi-vertex angle α approaches the value $\pi/2$. In this instance, the general nonlinear conical shell equations reduce to those for a circular flat plate, with a concentric hole, by virtue of another simple substitution; that is,

$$\alpha = \pi/2$$

$$x \sin \alpha = x = r \tag{108}$$

INPLANE STRAIN-DISPLACEMENT RELATIONS (rotations about the normal
neglected)

By virtue of equations (108), equations (11) reduce to

$$\epsilon_r = u,_r + \frac{1}{2} w,_r^2$$

$$\epsilon_\varphi = (u + v,_\varphi)/r + w,_\varphi^2/2r^2 \tag{109}$$

$$\gamma_{r\varphi} = v,_r + (u,_\varphi - v)/r + w,_r w,_\varphi/r$$

CURVATURE-DISPLACEMENT RELATIONS

Equations (12) reduce to

$$\varkappa_r = w,_{rr}$$

$$\varkappa_\varphi = W,_{\varphi\varphi}/r^2 + w,_r/r \tag{110}$$

$$\varkappa_{r\varphi} = W,_{r\varphi}/r - w,_\varphi/r^2$$

TOTAL POTENTIAL ENERGY

The total potential energy is given by

$$U_T = U_m + U_b + V_L + V_m + V_b \tag{111}$$

Reduction of equations (23) and (75)-(77) in conjunction with equation (111) yields

$$
\begin{aligned}
U_T = & \frac{Eh}{2(1-\nu^2)} \iint\limits_{\varphi\ r} \Big\{ (u,_r + \tfrac{1}{2} w,_r^2)^2 + [(u + v,_\varphi)/r + w,_\varphi^2/2r^2]^2 \\
& + 2\nu(u,_r + \tfrac{1}{2} w,_r^2)[(u + v,_\varphi)/r + w,_\varphi^2/2r^2] \\
& + \frac{1-\nu}{2} [v,_r + (u,_\varphi - v)/r + w,_r w,_\varphi/r]^2 \Big\} \ rdrd\varphi \\[6pt]
& + \frac{D}{2} \iint\limits_{\varphi\ r} \Big\{ w,_{rr}^2 + [w,_{\varphi\varphi}/r^2 + w,_r/r]^2 + 2\nu\, w,_{rr}[w,_{\varphi\varphi}/r^2 + w,_r/r] \\
& + 2(1-\nu)(w,_{r\varphi}/r - w,_\varphi/r^2)^2 \Big\} \ rdrd\varphi - \iint\limits_{\varphi\ r} (q_r u + q_\varphi v + q_z w)\ rdrd\varphi \\[6pt]
& - \int\limits_\varphi [r(\bar N_r u + \bar N_{r\varphi} v)]_{r_1}^{r_2}\, d\varphi - \int\limits_r [\bar N_\varphi v + \bar N_{r\varphi} u]_{\varphi_1}^{\varphi_2}\, dr \\[6pt]
& - \int\limits_\varphi [r(-\bar M_r w,_r + \bar M_{r\varphi} w,_\varphi/r + \bar Q_r w)]_{r_1}^{r_2}\, d\varphi - \int\limits_r [-\bar M_\varphi w,_\varphi/r \\[6pt]
& + \bar M_{r\varphi} w,_\varphi/r + \bar Q_\varphi w]_{\varphi_1}^{\varphi_2}\, dr \tag{112}
\end{aligned}
$$

FORCE-DISPLACEMENT EQUATIONS OF MOTION

Reduction of equations (41)-(43), with the addition of the applied load terms, yields

$$N_{r,r} + (N_r - N_\varphi)/r + N_{r\varphi,\varphi}/r = \rho h \ddot u - q_r$$

(continued)

$$(N_{\phi,\phi} + 2N_{r\phi})/r + N_{r\phi,r} = \rho h \ddot{v} - q_\phi \tag{113}$$

$$D\nabla_r^4 w - (1/r)[(w,_\phi N_{r\phi} + rw,_r N_r),_r + (w,_\phi N_\phi/r + w,_r N_{r\phi}),_\phi]$$

$$= -\rho h \ddot{w} + q_z$$

where

$$\nabla_r^4(\ ) = \nabla_r^2[\nabla_r^2(\ )] = \nabla_r^2[(\ ),_{rr} + (\ ),_r/r + (\ ),_{\phi\phi}/r^2] \tag{114}$$

BOUNDARY CONDITIONS

Equations (49)-(51) reduce to

(a) along $r = r_1$ and $r = r_2$

$$N_r = \bar{N}_r \qquad\qquad \underline{or} \quad \delta u = 0$$

$$N_{r\phi} = \bar{N}_{r\phi} \qquad\qquad \underline{or} \quad \delta v = 0$$

$$M_r + rM_{r,r} - M_\phi - 2M_{r\phi,\phi} \tag{115}$$

$$+ N_{r\phi} w,_\phi + rw,_r N_r = r\bar{Q}_r - \bar{M}_{r\phi,\phi} \qquad \underline{or} \quad \delta w = 0$$

$$M_r = \bar{M}_r \qquad\qquad \underline{or} \quad \delta w,_r = 0$$

(b) along $\phi = \phi_1$ and $\phi = \phi_2$

$$N_{r\phi} = \bar{N}_{r\phi} \qquad\qquad \underline{or} \quad \delta u = 0$$

$$N_\phi = \bar{N}_\phi \qquad\qquad \underline{or} \quad \delta v = 0$$

$$M_{\phi,\phi}/r - 2M_{r\phi}/r - 2M_{r\phi,r} \tag{116}$$

$$+ w,_\phi N_\phi/r + w,_r N_{r\phi} = \bar{Q}_\phi - \bar{M}_{r\phi,r} \qquad \underline{or} \quad \delta w = 0$$

$$M_\phi = \bar{M}_\phi \qquad\qquad \underline{or} \quad \delta w,_\phi = 0$$

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(c) at the corners of the segment

$$M_{r\varphi} = \bar{M}_{r\varphi} \tag{117}$$

CLASSICAL NONLINEAR EQUATIONS

Equations (58) reduce to

$$rN_{r,r} + N_r \cdot N_\varphi + N_{r\varphi,\varphi} = 0$$

$$N_{\varphi,\varphi} + 2N_{r\varphi} + rN_{r\varphi,r} = 0 \tag{118}$$

$$D\nabla_r^4 w - (1/r)[w,_\varphi N_{r\varphi} + rw,_r N_r),_r + (w,_\varphi N_\varphi/r + w,_r N_{r\varphi}),_\varphi] = -\rho h \ddot{w}$$

Using the first two of equations (118), the third may also be written as

$$D\nabla_r^4 w - N_\varphi(w,_x + w,_{\varphi\varphi}/r^2) - w,_{rr}\ddot{}_x + 2N_{x\varphi}(w,_\varphi/r^2 - w,_{r\varphi}/r) = -\rho h \ddot{w} \tag{119}$$

The stress function definition of equations (59) reduces to

$$N_r/h = F,_r/r + F,_{\varphi\varphi}/r^2$$

$$N_\varphi/h = F,_{rr} \tag{120}$$

$$N_{x\varphi}/h = F,_{r\varphi}/r - F,_\varphi/r^2$$

The compatibility relationship [equation (61)] reduces to

$$\nabla_r^4 F = (E/r^2)[w,_{r\varphi}^2 - w,_{rr} w,_{\varphi\varphi} - (w,_\varphi/r),_r - rw,_r w,_{rr}] \tag{121}$$

and the lateral equilibrium equation [equation (61)] reduces to

$$D\nabla_r^4 w - (h/r^2)[w,_{rr} F,_{\varphi\varphi} - 2w,_{r\varphi} F,_{r\varphi} + w,_{\varphi\varphi} F,_{rr} + 2(w,_\varphi F,_\varphi),_r/r$$

$$+ r(w,_r F,_\varphi),_r/r + r(w,_r F,_r),_r - 2w,_\varphi F,_\varphi/r^2] = -\rho h \ddot{w} \tag{122}$$

Equations (121) and (122) describe the nonaxisymmetric, nonlinear vibrational behavior of a thin circular plate. When axial symmetry

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is assumed, all derivatives with respect to φ vanish and the governing equations become

$$\nabla_r^4 F = -E w,_r w,_{rr}/r$$

$$D\nabla_r^4 w = h(w,_r F,_r),_r/r - \rho h \ddot{w}$$

(123)

These simplified equations are equivalent to those used in some recent studies of the nonlinear dynamical behavior of flat circular shells, notably those by Yamaki[26] and Bauer[27].

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APPENDIX V

SUGGESTED NUMERICAL SOLUTION PROCEDURE WITH ILLUSTRATIVE
APPLICATION TO THE NONLINEAR VIBRATIONS OF A BEAM

Additional definition of symbols:

A	cross-sectional area of beam
I	cross-sectional area moment of inertia
ℓ,m,i,r	integers
L	length of beam
M_{ri}	generalized mass functions
q_i	generalized coordinates
u	axial displacement of beam
w	transverse displacement of beam
x	axial coordinate of beam
z	transverse coordinate of beam
ρ	mass density of beam
$\Phi_i(x)$	functions of x
ψ_i	functions defined by equations (130)

The proposed numerical solution for the nonlinear vibrational behavior
of a conical shell can best be explained by demonstrating its use for
the simple case of the nonlinear vibration of a beam.

The expression for the total strain energy of a beam in the absence of
applied loads may be obtained directly from its conical shell counter-
part [equation (23)] as

$$U = \frac{1}{2}\int_o^L EA(u,_x + \frac{1}{2} w,_x^2)^2 dx + \frac{1}{2}\int_o^L EIw,_{xx}^2 \, dx \qquad (124)$$

and the kinetic energy is

$$T = \frac{1}{2}\int_o^L \rho A(\dot{u}^2 + \dot{w}^2) \, dx \qquad (125)$$

General displacement functions may be assumed in the form

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$$w = \sum_{i=1}^{\ell} \Phi_i(x) \, q_i(t)$$

$$u = \sum_{i=\ell+1}^{m} \Phi_i(x) \, q_i(t)$$

(126)

Substitution of equations (126) into equations (124) and (125) and the enforcement of the simultaneous vanishing of the first variation of the Lagrangian (T-U) with respect to the various $q_i(t)$ lead to a set of m second-order nonlinear differential equations; that is,

$$\sum_{i=1}^{\ell} M_{ri} \, \ddot{q}_i + \psi_r = 0 \qquad\qquad 1 \leq r \leq \ell$$

$$\sum_{i=\ell+1}^{m} M_{ri} \, \ddot{q}_i + \psi_r = 0 \qquad\qquad \ell+1 \leq r \leq m$$

(127)

where

$$M_{ri} = \int_0^L \rho A \, \Phi_i(x) \, \Phi_r(x) \, dx$$

(128)

and

$$\psi_r = \frac{\partial U}{\partial q_r} = \int_0^L EA \left[\sum_{i=\ell+1}^{m} \Phi_i'(x) q_i(t) \right] \left[\sum_{i=1}^{\ell} \Phi_i'(x) q_i(t) \right] \Phi_r'(x) \, dx$$

$$+ \frac{1}{2} \int_0^L EA \left[\sum_{i=1}^{\ell} \Phi_i'(x) q_i(t) \right]^3 \Phi_r'(x) \, dx$$

$$+ \int_0^L EI \Phi_r'' \sum_{i=1}^{\ell} \Phi_i''(x) q_i(t) dx \qquad\qquad (1 \leq r \leq \ell)$$

(129)

$$\psi_r = \frac{1}{2} \int_0^L EA \left[\sum_{i=1}^{\ell} \Phi_i'(x) q_i(t) \right]^2 \Phi_r'(x) \, dx$$

(continued)

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$$+ \int_0^L EA\Phi_r'(x) \sum_{i=l+1}^{m} \Phi_i'(x) q_i(t) dx \qquad (l+1 \le r \le m)$$

The two nonlinear second-order differential equations given by equations (127) may be reduced to $2m$ first-order nonlinear equations by the substitutions

$$\dot{q}_i = Q_i \qquad i = 1, 2, \ldots .m \qquad (130)$$

Equations (127) then become

$$\sum_{i=1}^{l} M_{ri} \dot{Q}_i + \psi_r = 0 \qquad 1 \le r \le l$$

$$\sum_{i=l+1}^{m} M_{ri} \dot{Q}_i + \psi_r = 0 \qquad l+1 \le r \le m \qquad (131)$$

Before the numerical solution is obtained, functions $\Phi_i(x)$, i=1, 2, ...m must be assumed. These may be trigonometric or polynomial in nature, or in fact, any type of function which satisfies the geometric boundary conditions of the problem. Once these functions are selected, the problem is given initial values

$$q_i(0) = q_{o_i}$$

$$\dot{q}_i(0) = \dot{q}_{o_i} \qquad (132)$$

and the solutions are obtained numerically using a greatly modified Runga-Kutta technique introduced by Rosser[30] and slightly modified further by Skappel[31].

For a given time increment ∇t, the initial values of q_i and $\dot{q}_i$ are used and the ψ_r functions are computed through numerical integration over x per iteration. This process continues until the required history is obtained. The order of approximation l and m must, of course, be given.

The main merit of this proposed numerical approach is that there
is no need to expand the assumed displacement functions by
multiplying out the infinite series and collecting terms in the various
q_i 's. This problem is prohibitively severe, especially in the case of
nonlinear problems wherein infinite series must be raised to the third
power such as evidenced in equation (129). This is precisely
why the bulk of nonlinear structural analyses to date are limited to
low-order approximations; in the case of the nonlinear vibrations of
thin shells, such low-order approximations do not allow for the devel-
opment of nonperiodic motion as demonstrated by Mayers and Wrenn[18]. It
is also evident from equations (129) that the method is not limited to
homogeneous beams, since the numerical process need not discern any
type of continuity in the axial direction.

With the use of the numerical approach proposed here, the order of the
approximation (that is, the number of terms retained in the series) may
be selected at will and limited convergence tests may be run, the
limitation being the amount and expense of computer time budgeted.
Finally, the numerical process itself is known to be smooth, and no
numerical roundoff difficulties are anticipated.

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13. ABSTRACT

Equations of motion and associated boundary conditions are developed for the general nonlinear vibrational behavior of thin conical shells. The theory is based upon nonlinear strain-displacement relations deduced for a conical shell from those derived by Sanders for thin shells of compound curvature. Equations for the bending, buckling, and postbuckling of conical shells under arbitrary loads are also developed and are shown to reduce to equations based on more simplified theories for both conical and circular cylindrical shells and circular flat plates. Various solution approaches to the nonlinear conical shell vibration problem are examined, and a new numerical method of solution is proposed and discussed.

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